Calculus 1, Week 15

Hey Calculus tutors and students! Welcome to the week 15 resource for Calculus 1. This will be the last resource for Calculus 1 on the semester. In this resource I will go back over some of the most important topics to help prepare you for the cumulative final. Unfortunately, I do not have the room to go over the entire class, so I am not able to go through every topic in detail. In addition to this final resource, there are also the previous week's resources, if you want a more in depth discussion of a particular topic. Also, there will be one last calculus 1 group tutoring session this Wednesday at 5pm.

I hope that the Calc 1 resources I shared with you this semester were helpful and efficient in addressing your academic needs. As we approach the end of the fall semester, I would like to let you know that all resources for this semester can be found here: https://www.baylor.edu/support_programs/index.php?id=967950

As with last year, if anyone has any suggestions for things to add to the resource please do not hesitate to contact me!

- Alex Williamson

Key

- Yellow Highlighting: Definitions that you need to know.
- Green Highlighting: Explanation of how you actually go about doing the problems.
- Blue Highlighting: Helpful insight into why certain definitions work, how to think about problems, etc.

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

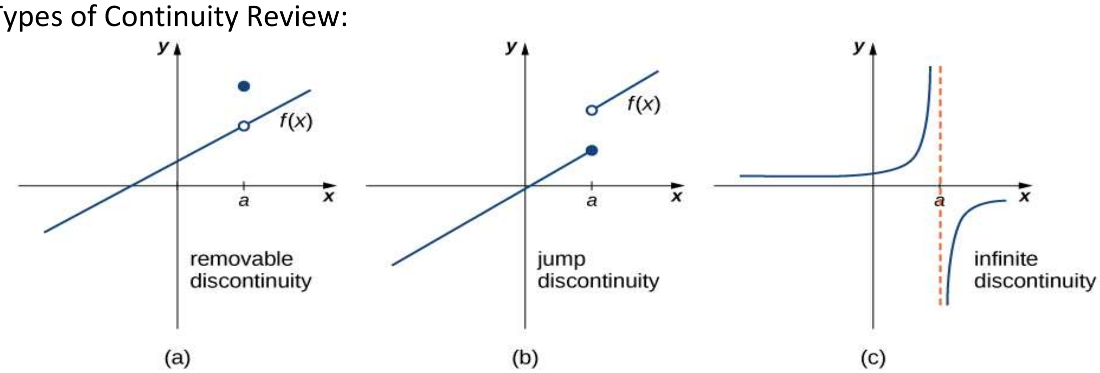
$$= \lim_{x \to a} \frac{f(x) - f(a)}{x \to a}$$

Calcworkshop.com

Concepts

Limits (covered in the resources for week 3 and week 4)

- Types of Continuity Review:



Source: https://math.libretexts.org/Courses/Misericordia_University/MTH_171-172%3A_Calculus_-_Early_Transcendentals_(Stewart)/02%3A_Limits_and_Derivatives/2.05%3A_Continuity

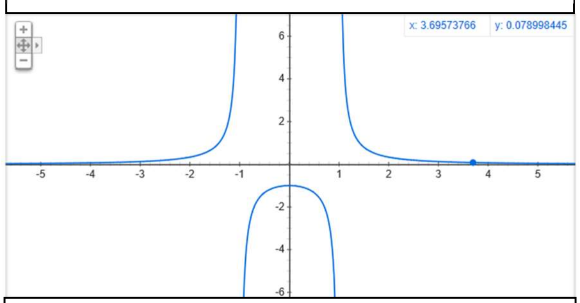
- Squeeze Theorem Review: $If\ g(x) \leq f(x) \leq h(x), then\ \lim_{x->N} g(x) \leq \lim_{x->N} f(x) \leq \lim_{x->N} h(x)$
- Example Problems
    - Find the derivative of $f(x) = \sqrt{x^4 - 4}$ using the limit definition of the derivative, then check your work by taking the derivative normally.
        - A: $f\,'(x) = \frac{1}{2}(x^3 - 4)^{-1/2}(4x^3)$
    - What is $\lim_{x\to 2} \left(\frac{x-2}{x^2-4}\right)^{1/2}$
        - A: $\frac{1}{2}$
    - What is the $\lim_{x\to 0} x\,sin\left(\frac{5^{2x}}{3x^4-243}\right)$ using the squeeze theorem?
        - A: 0

Derivatives (covered in the resources for weeks 4-10)

- Related Rates Review: Step-by-step process
    - 1st step: draw a picture with everything you know on it.
    - 2nd step: Identify what the question is asking you for (for example, how fast the ladder is sliding down the wall after three seconds) and translate it into math notation (for example, $\frac{dh}{dt}\ at\ t = 3$).
    - 3rd step: use the picture and your knowledge of geometry to create an equation that relates all the pieces that you have to the thing you are looking for.
    - 4th step: solve the equation you have created.
- Linear Approximation Review: The linear approximation of a function is defined as $L(x) = f(a) + f'(a)(x - a)$, with $L(x) \approx f(x)$
    - In differential notation we treat $\Delta y \approx dy$ and $\Delta x \approx dx$ for a small dy and dx.
    - The error and percentage error for of our Linear Approximation is defined as
        - $error = |\Delta f - f'(a)\Delta x|$
        - $\%\ error = \left|\frac{error}{actual\ value}\right| * 100\%$
- Example Problems
    - What is the derivative of f(x) = $\frac{3x^5}{lnx}$?
        - f'(x) = $\frac{lnx*(3*5x^5) - 3x^5*\frac{1}{x}}{(lnx)^2}$
    - What is $\frac{d}{dx}(\cos(x))^2$?
        - $2(\cos(x)) * (-\sin(x)) * \frac{dx}{dx}$
    - Solve for $\frac{dy}{dx}\ if\ y^7 + ln(x) = 27$
        - $\frac{dy}{dx} = \frac{-1}{x} * \frac{1}{7y^6}$
    - What is the derivative of $\arcsin 3x^4$

    - $\frac{1}{\sqrt{1-9x^8}} * 3 * 4x^3$
- What is the derivative of $\log_3 x^7$?
    - $\frac{7}{\ln(3)*x}$
- A ladder is against a building and slowly sliding down. Let h(t) be the height of the ladder at time t, let x(t) be the distance between the base of the building and the ladder at time t, the ladder is 10 meters long, x(0)=2, and the speed at which the base of the ladder is moving away from the building is 2 m/s. Find the speed at which the height of the ladder (h(x)) is changing at t=2.
    - Answer: $\frac{dh}{dt} = -1.5 \text{ m/s}$
- Estimate $\sqrt[3]{x}$ from the point x=8 to the point x=8.12. What is the % error of your estimation?
    - $L(8.12) = 2 + (1/12)(0.12) = 2.01$
    - $\%\ error = \left|\frac{2.00995 \quad .01}{2.00996}\right| = .003\%$
- Sketch the graph of $\frac{1}{x^2-1}$
    - Answer: Graph 1
- Sketch the graph of $\frac{-x^2}{x^2+1}$
    - Answer: Graph 2
- Sketch the graph of $e^x x^4$
    - Answer: Graph 3

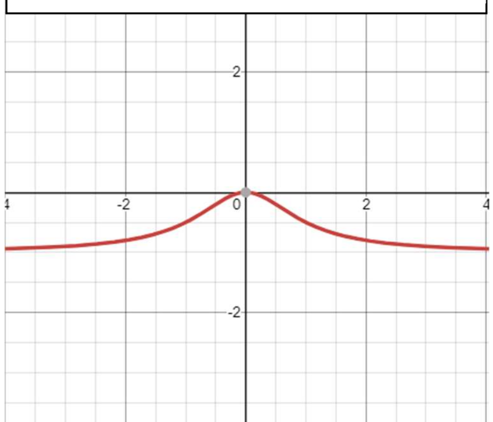
Graph 1



Source: Google Calculator

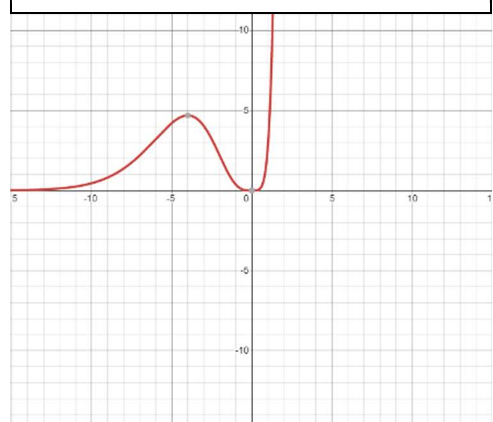
https://www.google.com/search?q=graph+of+1/%28x%5E2-1%29&source=lmns&bih=967&biw=1920&safe=active&hl=en&ved=2ahUKEwjc7ICCo7PoAhWSkZ4KHSOTAbcQ_AUoAHoECAEQAA&safe=active

Graph 2



Source: https://www.desmos.com/calculator

Graph 3



Source: https://www.desmos.com/calculator

- Minimize the cost of producing a box with a volume of $100m^3$, given that the base of a box is a square, the cost per square meter of the sides of the box is \$1, and the cost per square meter of the top and bottom of the box is \$2. What is the height of the box?
    - A: $h \approx 7.37m$

Integrals (covered in the resources for weeks 11-13)

- Fundamental Theorem of Calculus review:
    - The 1st part of the fundamental theorem of calculus states that $\int_a^b f(x)dx = F(b) - F(a)$, where F(x) is the anti-derivative of f(x).
    - The second part of the fundamental theorem of calculus states that $\frac{d}{dx}\int_a^x f(t)dt = f(x)$. Also, as an implication, if $G(x) = A\big(g(x)\big) = \int_a^{g(x)} f(t)dt$ $then\ G'(x) = A'(g(x)) * (g'(x)) = g'(x)f(x)$
- U-Substitution Review: U-sub works by replacing the complicated part of an integral with a variable "u," which we use to make the problem much simpler. Mathematically, we are changing the integral from being $\int f\big(u(x)\big)u'(x)dx$ to looking like $\int f(u)du$. If our integral is indefinite, we will need to switch back from being in terms of "u" at the end of our problem. On the other hand, if our integral is definite, then we will also need to change the bounds of integration from being "a" and "b" to being "u(a)" and "u(b)."

Another way to write U-substitution

$$\int f(g(x))g'(x)dx = \int f(u)du = F(u) + C$$

where $u = g(x)$ $du = g'(x)$

Source: https://calcworkshop.com/integrals/u-substitution/

- Example Problems
    - $\int_1^7 x^5 dx$
        - $\frac{(7)^6-(1)^6}{6}$
    - Find upper and lower bounds for $\int_0^5 \cos(x)dx$
        - $\int_0^5 (-1)dx \le \int_0^5 \cos(x)dx \le \int_0^5 (1)dx$

$$-5 \le \int_0^5 \cos(x)dx \le 5$$

    - $\int (x^5 + \frac{1}{x})dx$
        - $\frac{x^6}{6} + ln|x| + c$
    - $\int_{-1}^5 sec(x)tan(x)dx$
        - $sec(5) - \sec(-1)$
    - What is the derivate of $G(x) = \int_1^{Ln(x)} sin(t)\, dt$
        - $G(x) = A(Ln(x)), where\ A(x) = \int_1^x sin(t)dt$, so $G'(x) = A'(Ln(x)) * \left(\frac{1}{x}\right)$. Since $A(x) = \int_1^x sin(t)dt$, $A'(Ln(x)) = sin(Ln(x))$. Thus, $G'(x) = sin(Ln(x)) * \left(\frac{1}{x}\right)$.
    - What was the total displacement of a person moving at velocity $= t^3 - 9t$ from t=1 to t=7? What was the distance?

- Displacement $= \int_1^7 (t^3 - 9t)dt = 384$
- Distance $= \int_1^7 |(t^3 - 9t)|dt = -\int_1^3 (t^3 - 9t)dt + \int_3^7 (t^3 - 9t)dt = 416$
    - The function $t^3 - 9t$ is negative from [1,3], so I had to split the integral into two pieces and manually ensure that both pieces would be positive.
- Follow up question: in the above example we got a larger distance than displacement. Does this make sense? Is there any time that displacement would be greater than distance?
    - A: It does make sense, and distance is always greater than or equal to displacement.

- What was the total displacement of a person moving at velocity $= t^5 - 8t^2$ from t=0 to t=6? What was the distance?
    - Displacement $= \int_{-5}^{5} (t^5 - 8t^2)dt = -666.6666666666667$
        - Does it make sense that we would have a negative displacement?
            - Yes – displacement can be negative. It is distance that is always positive.
    - Distance $= \int_{-5}^{5} |(t^5 - 8t^2)|dt = -\int_{-5}^{2} (t^5 - 8t^2)dt + \int_2^5 (t^5 - 8t^2)dt =$ 5229.667
- $\int_{-3}^{7} x \sec^2(x^2 - 4)\, dx$
    - $Set\ u(x) = x^2 - 4\ and\ du = 2xdx.$
    $As\ such\ we\ can\ rewrite\ the\ integral\ as\ \int_{u(-3)}^{u(7)} \sec^2 u * \frac{du}{2}.$
    $\int_{u(-3)}^{u(7)} \sec^2 u * \frac{du}{2} = \frac{1}{2}\tan(u(7)) - (\frac{1}{2}) \tan(u(-3)) = \frac{1}{2}\tan(45) + \frac{1}{2}\tan(5)$
- $\int_1^5 (x^2 + x)\, e^{x^3+3x}\, dx$
    - $Set\ u(x) = x^3 + 3x\ and\ du = (3x^2 + 3x)dx.$
    $As\ such\ we\ can\ rewrite\ the\ integral\ as\ \int_{u(1)}^{u(5)} e^u * \frac{du}{3}$
    $\int_{u(1)}^{u(5)} e^u * \frac{du}{3} = \frac{1}{3}e^{u(5)} - \left(\frac{1}{3}\right) e^{u(1)} = \frac{1}{3}e^{90} + \frac{1}{3}e^6$
- $\int_{-5}^{6} \frac{1}{x} sec(Ln(x))\, tan(Ln(x))dx$
    - $Set\ u(x) = Ln(x)\ and\ du = \frac{1}{x}dx. As\ such$
    $we\ can\ rewrite\ the\ integral\ as\ \int_{u(-5)}^{u(6)} sec(u)tan(u) * du$
    $\int_{u(-5)}^{u(6)} sec(u)tan(u) * du = sec(u(6)) - sec(u(-5)) = sec(Ln(6)) - sec(Ln(-5))$
- $\int_3^9 (x^3 + x)(x^4 + 2x^2)^3 dx$
    - $Set\ u(x) = x^4 + 2x^2\ and\ du = (4x^3 + 4)dx.$

$As\ such\ we\ can\ rewrite\ the\ integral\ as\ \int_{u(3)}^{u(9)} u^3 * \frac{du}{4}\ \int_{u(3)}^{u(9)} u^3 * \frac{du}{4} = \frac{1}{4*4}u(9)^4 - \frac{1}{4*4}u(3)^4 = \frac{1}{16}(6885)^4 - \frac{1}{12}(153)^4$

- Solve $\int \frac{xd}{x^4+1}$
  - $\int \frac{xdx}{x^4+16} = \int \frac{xdx}{\left(\frac{x^4}{16}\right)+1}$

    set u= $\frac{1}{4}x^2\ and\ du = \frac{1}{2}xdx$

    $2\int \frac{du}{u^2+1} = 2\tan^{-1} u + c = 2\tan^{-1}(\frac{1}{4}x^2) + c$

- Solve $\int \frac{\sqrt{x}dx}{\sqrt{1-5(x^3)}}$
  - $\int \frac{xdx}{\sqrt{1-5(x^3)}} = \int \frac{xdx}{\sqrt{1-(\sqrt{5}x^{\frac{3}{2}})^2}}$

    set u= $\sqrt{5}x^{\frac{3}{2}}\ and\ du = \frac{3\sqrt{5}}{2}\sqrt{x}dx$

    $\frac{2}{3\sqrt{5}}\int \frac{du}{\sqrt{1-u^2}} = \frac{2}{3\sqrt{5}}\sin^{-1} u + c = \frac{2}{3\sqrt{5}}\sin^{-1} \sqrt{5}x^{\frac{3}{2}} + c$

- Solve $\int \frac{x^2dx}{|6x^3|\sqrt{36(x^6)-1}}$
  - $\int \frac{x^3dx}{|6x^3|\sqrt{(6x^3)^2-1}}$

    set u= $6x^3\ and\ du = 18x^2dx$

    $$\frac{1}{18}\int \frac{du}{|u|\sqrt{(u)^2-1}} = \frac{1}{18}\sec^{-1} u + c = \frac{1}{18}\sec^{-1}(6x^3) + c$$