MTH 1322: Calculus II
Week 14 Tutoring Resources

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Welcome Calculus II tutors and students! In this week's resource we will be continuing our work with infinite series. We will go over new ways to determine convergence for different types of series. For more help with these topics please schedule a 1-on-1 visit with me or another tutor. **Please visit baylor.edu/tutoring to make an appointment and to reserve a spot for the Calculus II group tutoring session every Tuesday at 6:30pm.** If you would like to view any of the previous resources please click **HERE**.

Overview[1]

KEYWORDS: Power Series / Taylor Series

1 New Topics

1.1 Power Series

For this resource we look at special kinds of infinite series called power series. We define power series to be the series:

$$F(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n \tag{1}$$

Where we define c to be the center of the series. Recall that every power series has a radius of convergence R where R is either a finite number or infinity. We say that F(x) converges absolutely when $|x-c| < R$ and diverges when $|x-c| > R$. The natural follow up question after reading the previous statement is, how do we determine R. As it turns out we already know how to do find R. Recall then we learned about ration and root test we touched on this matter. We had said that ratio test:

$$\rho = \lim_{n\longrightarrow\infty}\left|\frac{a_{n+1}}{a_n}\right| \tag{2}$$

By previous results we know that $\rho = 1/R$ or alternatively we can say that $R = 1/\rho$. Similarly for the root test we can also determine R. The root test:

$$L = \lim_{n\longrightarrow\infty} \sqrt[n]{|a_n|} \tag{3}$$

Yet again the results are almost identical. We can say that $L = 1/R$ or alternatively we can say $R = 1/L$. As you can see the results we get from the ratio a root test inversely related to the radius of convergence. For example if we found the ratio test yielded a result of $\rho = \infty$ we can say that the radius of convergence is 0 since $1/\infty = 0$. Another quick example, if found that the root test yielded a result of $L = 0$ then we can say the radius of convergence is equal to ∞ since $1/0 = \infty$. One last important note is that when dealing

[1] The information used to create this resource was taken from this source: [1]

with power series we only need to consider the coefficient a_n and not the $(x-c)^n$ portion. Therefore, as we will see in our example below, we only take the ratio test of the coefficient.

Let's work the following example: consider the power series below and determine the radius of convergence.

$$\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^3} x^n \tag{4}$$

Our first step is to identify a_n. As we can see we find that $a_n = \frac{1}{n!}$ and therefore we need to determine a convergence test to use. We stated in the last resource that using ratio test is ideal for when we are dealing with factorials. Therefore if we apply ratio test to our a_n:

$$\rho = \lim_{n \longrightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \longrightarrow \infty} \left| \frac{(2n+2)!}{((n+1)!)^3} \cdot \frac{(n!)^3}{(2n)!} \right| = \lim_{n \longrightarrow \infty} \frac{(2n+2)(2n+1)2n!}{(n+1)^3(n!)^3} \cdot \frac{(n!)^3}{(2n)!} \tag{5}$$

If we look back to last weeks resource we worked an example that dealt with how to with $n!$ and $(n+1)!$. In this case we use that helpful trick to simplify the fraction before applying the limit.

$$\rho = \lim_{n \longrightarrow \infty} = \frac{(2n+2)(2n+1)}{(n+1)^3} = \lim_{n \longrightarrow \infty} \frac{n^2(2+\frac{2}{n})(2+\frac{1}{n})}{n^3(1+\frac{1}{n})^3} = \frac{1}{\infty} = 0 \tag{6}$$

Now our next step is to take the inverse of our result. Since we found that $\rho = 0$ it follows that $R = \frac{1}{0} = \infty$. Therefore, we can say that our power series centered at 0, has a radius of convergence equal to ∞.

Working with infinite series is consistently where students struggle the most so please do not hesitate for help, myself or another tutor will be glad to help. If you would like to watch a short video that works more with power series and radius convergence please click **HERE** [2]. If you have more time and would like to watch a longer video I recommend watching this **VIDEO**. [3]

1.2 Taylor Series

Taylor series, also known as Taylor polynomials, are special case of power series and ironically enough Maclaurin series are a special form of Taylor Series. The general form of Taylor series looks very similar that of power series. As we can see we define a Taylor series expansion of a function f to be:

$$T_n(x) = \sum_{n=0}^{\infty} a_n(x-a)^n \tag{7}$$

where a is the center of the series. For power series we let a_n be open for interpretation, but for Taylor Polynomials we must have:

$$a_n = \frac{f^n(a)}{n!} \tag{8}$$

Notice the f^n is the nth derivative of the expanded function f. Note: Maclaurin Series are Taylor Series centered at a = 0. For Taylor series we are not worries so much with convergence and divergence but rather we seek to express functions in the form of an infinite series.

Let's work an example. Suppose we are asked to find the Taylor series expansion of the following function.

$$f(x) = e^{2x} \quad a = 1 \tag{9}$$

To find the Taylor series expansion we need to first consider the power series form of e^x. It is important to note that it is recommended to have memorized e^x as well as $\cos x$ and $\sin x$ and perhaps a few other common ones to be safe. If we start with the power series expansion of e^{2x} we can see we need to follow these steps: we first start with examining the derivatives of e^2x. Notice that:

$$f'(1) = 2e^2, \quad f''(1) = 2^2e^2 \quad f'''(1) = 2^3e^3 \tag{10}$$

Thus we can a pattern develop. Now we can say that for our series:

$$a_n = \frac{f^n(a)}{n!} = \frac{2^ne^2}{n!} \tag{11}$$

For our next step we want to center our series at a = 1.

$$\sum_{n=0}^{\infty} a_n (x-1)^n = \sum_{n=0}^{\infty} \frac{2^n e^2}{n!}(x-1)^n \tag{12}$$

If we look closely we can see that this closely resembles the original power series expansion e^x.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{13}$$

Furthermore, if we were asked to find the radius of convergence for equation (12) we could do by using ration test. Be careful when working with these problems as it may require you evaluate the first few derivatives by hand so that you can see the clear pattern.

If you need more help working with series please schedule a 1-on-1 appointment with myself or another tutor. To watch a video that works another example of Taylor series please click **HERE** [4]. If you have time I also recommend watching this slightly longer video that works multiple examples of Taylor series which you find by clicking **HERE** [5].

$$4^n = \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} \tag{14}$$

References

[1] J. Rogawski, C. Adams, and R. Franzosa, *Calculus: Early Transcendentals*, 4th ed. New York: W. H. Freeman, Dec. 2018.

[2] Dr. Trefor Bazett, "Power Series & Intervals of Convergence," Feb. 2019. [Online]. Available: https://www.youtube.com/watch?v=XHoRBh4hQNU

[3] The Organic Chemistry Tutor, "Power Series - Finding The Radius & Interval of Convergence - Calculus 2," Apr. 2018. [Online]. Available: https://www.youtube.com/watch?v=EGni2-m5yxM

[4] blackpenredpen, "The Formula for Taylor Series," Jan. 2019. [Online]. Available: https://www.youtube.com/watch?v=0WHThuWxwx0

[5] The Organic Chemistry Tutor, "Taylor Series and Maclaurin Series - Calculus 2," Apr. 2018. [Online]. Available: https://www.youtube.com/watch?v=LDBnS4c7YbA