Calculus 1, Week 14

Hey Calculus tutors and students! This resource covers the fourteenth week of class. Because less material is covered this week, I will be including some extra review questions on last week’s topics. In addition, I will go over how to take integrals that become inverse trig functions – a topic introduced in 5.8.

In addition to this resource and our one-on-one tutoring sessions, the Baylor Success Center also runs a weekly group tutoring session for Calculus 1. The group tutoring session occurs every Wednesday from 5pm-6pm. If you are interested in attending you can reserve a spot through our website: https://www.baylor.edu/support_programs/index.php?id=40917

As with last year, if anyone has any suggestions for things to add to the resource please do not hesitate to contact me!

- Alex Williamson

Keywords: Fundamental Theorem of Calculus (2nd part), Distance, Displacement, U-substitution, Inverse Trig Integration.

Key

- **Yellow Highlighting**: Definitions that you need to know.
- **Green Highlighting**: Explanation of how you actually go about doing the problems.
- **Blue Highlighting**: Helpful insight into why certain definitions work, how to think about problems, etc.

Concepts

Chapter 5.5 Review

- Example Problems
  - What is the derivative of \( G(x) = \int_2^{\sqrt{x}} \cos(t) \, dt \)
    - \( G(x) = A(\sqrt{x}) \), where \( A(x) = \int_2^{x} \cos(t) \, dt \), so
      \[
      G'(x) = A'(\sqrt{x}) \times \left( \frac{1}{2\sqrt{x}} \right). \quad \text{Since} \quad A(x) = \int_2^{x} \cos(t) \, dt,
      \]  
      \[
      A'(\sqrt{x}) = \cos(\sqrt{x}). \quad \text{Thus} \quad G'(x) = \cos(\sqrt{x}) \times \left( \frac{1}{2\sqrt{x}} \right). 
      \]
  - What is the derivative of \( G(x) = \int_7^{5x^4} \sqrt{t} \, dt \)
    - \( G(x) = A(5x^4) \), where \( A(x) = \int_7^{x} \sqrt{t} \, dt \), so
      \[
      G'(x) = A'(5x^4) \times (20x^3). \quad \text{Since} \quad A(x) = \int_7^{x} \sqrt{t} \, dt, 
      \]
\[ A'(5x^4) = \frac{2}{3} (5x^4)^{3/2}. \text{ Thus, } G'(x) = \frac{2}{3} (5x^4)^{3/2} \ast (20x^3). \]

**Chapter 5.6 Review**

- **Example Problems**
  - What was the total displacement of a person moving at velocity \( t^3 - 9t \) from \( t=1 \) to \( t=7 \)? What was the distance?
    - Displacement = \( \int_1^7 (t^3 - 9t) \, dt = 384 \)
    - Distance = \( \int_1^7 \left| (t^3 - 9t) \right| \, dt = -\int_1^3 (t^3 - 9t) \, dt + \int_3^7 (t^3 - 9t) \, dt = 416 \)
    - The function \( t^3 - 9t \) is negative from \([1,3]\), so I had to split the integral into two pieces and manually ensure that both pieces would be positive.
    - Follow up question: in the above example we got a larger distance than displacement. Does this make sense? Is there any time that displacement would be greater than distance?
      - A: It does make sense, and distance is always greater than or equal to displacement.
  - What was the total displacement of a person moving at velocity \( t^2 - 8t \) from \( t=0 \) to \( t=6 \)? What was the distance?
    - Displacement = \( \int_0^6 (t^2 - 8t) \, dt = -63 \)
    - Does it make sense that we would have a negative displacement?
      - Yes – displacement can be negative. It is distance that is always positive.
    - Distance = \( \int_0^6 \left| (t^2 - 8t) \right| \, dt = -\int_0^8 (t^2 - 8t) \, dt + \int_8^6 (t^2 - 8t) \, dt = 583 \)

**Chapter 5.7 Review**

- **Example Problems**
  - \( \int_2^6 x^2 \cos x^3 \, dx \)
    - Set \( u(x) = x^3 \) and \( du = 3x^2 \, dx \).
      - As such we can rewrite the integral as \( \int_{u(2)}^{u(9)} \cos u \ast \frac{du}{3} \).
      - \( \int_{u(2)}^{u(9)} \cos u \ast \frac{du}{3} = \frac{1}{3} \sin(u(9)) - \left( \frac{1}{3} \right) \sin(u(2)) = \frac{1}{3} \sin(729) + \frac{1}{3} \cos(8) \)
  - \( \int_1^7 xe^{x^2-7} \, dx \)
    - Set \( u(x) = x^2 - 7 \) and \( du = 2x \, dx \).
      - As such we can rewrite the integral as \( \int_{u(1)}^{u(7)} e^u \ast \frac{du}{2} \)
      - \( \int_{u(1)}^{u(7)} e^u \ast \frac{du}{2} = \frac{1}{2}e^{u(7)} - \left( \frac{1}{2} \right)e^{u(1)} = \frac{1}{2}e^{42} + \frac{1}{2}e^6 \)
\[ \int_{-4}^{4} e^x \sec(e^x) \tan(e^x)dx \]

- Set \( u(x) = e^x \) and \( du = e^x dx \). As such we can rewrite the integral as
\[ \int_{u(-4)}^{u(4)} \sec(u) \tan(u) * du \]
\[ \int_{u(-4)}^{u(4)} \sec(u) \tan(u) * du = \sec(u(2)) - \sec(u(-4)) = \sec(e^2) - \sec(e^{-4}) \]

\[ \int_{0}^{10} (x^2 + 1)(x^3 + 3x^3)dx \]

- Set \( u(x) = x^3 + 3x \) and \( du = (3x^3 + 3)dx \). As such we can rewrite the integral as
\[ \int_{u(0)}^{u(10)} u^3 * \frac{du}{3} \]
\[ \int_{u(0)}^{u(10)} u^3 * \frac{du}{3} = \frac{1}{3^4}u(10)^4 - \frac{1}{3^4}u(0)^4 = \frac{1}{12}(1030)^4 - \frac{1}{12}(0)^4 \]

Chapter 5.8

- In chapter 5.8 the students learn how to integrate a few more functions. Specifically, they learn
  
  - \( \int \frac{dx}{x} = \ln x + c \)
  - \( \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c \)
  - \( \int \frac{dx}{x^2+1} = \tan^{-1} x + c \)
  - \( \int \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1} x + c \)

- As it turns out, due to the equation of inverse cos, cot, and csc, the integrals that would produce these functions are identical to the three inverse trig functions above, except with a negative sign in front of the integral. As such, you don’t need memorize them independently as long as you can remember how they are similar to the definitions above.

- New Video Resource
  - [https://www.youtube.com/watch?v=CFmTDn7BQmE](https://www.youtube.com/watch?v=CFmTDn7BQmE)

- Example Problems
  
  - Solve \( \int \frac{x^2dx}{x^4+9} \)
    
    \[ \int \frac{x^2dx}{x^4+9} = \int \frac{x^2}{(\frac{x^2}{3})+1} \]
    
    set \( u=\frac{1}{3}x^2 \) and \( du = \frac{2}{3}xdx \)
    
    \[ \frac{3}{2} \int \frac{du}{u^2+1} = \frac{3}{2}\tan^{-1} u + c = \frac{3}{2}\tan^{-1}(\frac{1}{3}x^2) + c \]

  - Solve \( \int \frac{x^2dx}{\sqrt{1-5(x^6)}} \)
    
    \[ \int \frac{x^2dx}{\sqrt{1-5(x^6)}} = \int \frac{x^2dx}{\sqrt{1-(\sqrt{5}x^3)^2}} \]
    
    set \( u=\sqrt{5}x^3 \) and \( du = 3\sqrt{5}x^2dx \)
\[
\frac{1}{3\sqrt{5}} \int \frac{du}{\sqrt{1-u^2}} = \frac{1}{3\sqrt{5}} \sin^{-1} u + c = \frac{1}{3\sqrt{5}} \sin^{-1} \sqrt{5}x^3 + c
\]

- Solve \( \int \frac{x}{x^3/2} \)
  - \( \int \frac{x}{x^3/2} \)
    - set \( u = x^{3/2} \) and \( du = \left( \frac{3}{2} \right)x^{1/2} \) \( dx \)
    - \( \frac{2}{3} \int \frac{du}{u} = \frac{2}{3} \ln(u) + c = \frac{2}{3} \ln(x^{3/2}) + c \)

- Solve \( \int \frac{x^3}{\sqrt{7}x^4\sqrt{\sqrt{7}x^2-1}} \)
  - \( \int \frac{x^3}{\sqrt{7}x^4\sqrt{\sqrt{7}x^2-1}} \)
    - set \( u = \sqrt{7}x^4 \) and \( du = 4\sqrt{7}x^3 \) \( dx \)
    - \( \frac{1}{4\sqrt{7}} \int \frac{du}{|u|\sqrt{(u)^2-1}} = \frac{1}{4\sqrt{7}} \sec^{-1} u + c = \frac{1}{4\sqrt{7}} \sec^{-1}(\sqrt{7}x^4) + c \)

- Solve \( \int \frac{x^3}{x^8+25} \)
  - \( \int \frac{x^3}{x^8+25} = \int \frac{x^3}{\left( \frac{x^8}{25} \right)+1} \)
    - set \( u = \frac{1}{5} x^4 \) and \( du = \frac{4}{5} x^3 \) \( dx \)
    - \( \frac{5}{4} \int \frac{du}{u^2+1} = \frac{5}{4} \tan^{-1} u + c = \frac{5}{4} \tan^{-1}(\frac{1}{5} x^4) + c \)

- Solve \( \int \frac{x}{\sqrt{1-7(x^4)}} \)
  - \( \int \frac{x}{\sqrt{1-7(x^4)}} = \int \frac{x}{\sqrt{1-(\sqrt{7}x^2)^2}} \)
    - set \( u = \sqrt{7}x^2 \) and \( du = 2\sqrt{7}x \) \( dx \)
    - \( \frac{1}{2\sqrt{7}} \int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2\sqrt{7}} \sin^{-1} u + c = \frac{1}{2\sqrt{7}} \sin^{-1} \sqrt{7}x^2 + c \)

- Solve \( \int \frac{x^3}{\sqrt{7}x^4\sqrt{16(x^2)-1}} \)
  - \( \int \frac{x^3}{\sqrt{7}x^4\sqrt{16(x^2)-1}} \)
    - set \( u = 4x \) and \( du = 4 \) \( dx \)
    - \( \frac{1}{4} \int \frac{du}{|u|\sqrt{(u)^2-1}} = \frac{1}{4} \sec^{-1} u + c = \frac{1}{4} \sec^{-1}(4x) + c \)
Things Students Tend to Struggle With

- **Integrals of Inverse Trig Functions**
  - Unfortunately, the integrals that become inverse trig functions are just one of the things in Calculus 1 that you end up having to memorize. One nice thing about them is that they look very similar to one another – for instance, the derivatives of arcsin and arccos are the same except for a negative sign. Similarly, they all appear as fractions, and most of them have radicals in the denominator. So if you see a problem that requires you to integrate a fraction, one of the first things you should do is consider if your integral looks like it might become an inverse trig function.