MTH 1322: Calculus II
Week 13 Tutoring Resources

Ethan Reyes

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Welcome Calculus II tutors and students! In this week's resource we will be continuing our work with infinite series. We will go over new ways to determine convergence for different types of series. For more help with these topics please schedule a 1-on-1 visit with me or another tutor. **Please visit baylor.edu/tutoring to make an appointment and to reserve a spot for the Calculus II group tutoring session every Tuesday at 6:30pm.** If you would like to view any of the previous resources please click **HERE**.

Overview[1]

KEYWORDS: Ratio Test / Root Test

1 New Topics

1.1 Ratio Test

This weeks resource we will review how to work two common tests for convergence. The first test we review is Ratio test. Ratio test is very useful for series with factorials as part of the coefficient. Suppose we are asked to find if the following series converges.

$$\sum_{n=0}^{\infty} a_n \tag{1}$$

We can use ratio test on the series to determine convergence. Recall that we need the following limit to exist to use the ratio test.

$$\rho = \lim_{n \longrightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \tag{2}$$

Therefore if we have find that $\rho < 1$ then $\sum a_n$ converges absolutely. Notice that because we take the absolute value of the ratio we are able to conclude the series converges absolutely. However, if we find that $\rho > 1$ then $\sum a_n$ diverges. Lastly, if we know that $\rho = 1$ then the test is inconclusive. Notice the similarity to the divergence test: if the limit of a_n was equal to 0 then we cannot make a conclusion about the series.

Let's work an example. Like we stated earlier, ratio test is very useful when dealing factorial in a series. Suppose we asked to determine if the following series converges or not.

$$\sum_{n=0}^{\infty} \frac{(n!)^3}{(3n+1)!} \tag{3}$$

Since we that we have a factorial in the series it seems reasonable to first use the ratio test to determine convergence. To start we only consider a_n. For our example we let $a_n = (n!)^3/(3n+1)!$. Our next step is to

[1] The information used to create this resource was taken from this source: [1]

find a_{n+1}, which we find to be $((n+1)!)^3/(3(n+1)+1)!$. Then we want to take the limit of the absolute value of $\frac{a_{n+1}}{a_n}$.

$$\rho = \lim_{n\longrightarrow\infty}\left|\frac{((n+1)!)^3}{(3(n+1)+1)!} \div \frac{(n!)^3}{(3n+1)!}\right| = \lim_{n\longrightarrow\infty}\left|\frac{((n+1)!)^3}{(3(n+1)+1)!}\cdot\frac{(3n)!}{(n!)^3}\right| \tag{4}$$

Notice that because we dividing two fractions we can multiply by the reciprocal. Furthermore, we also see that all of our terms are positive so we can drop the absolute value. For our next step we need to make use of our knowledge of factorials. Recall that by definition $n!$ is equal to $n \cdot (n-1) \cdot (n-2) \ldots (3)(2)(1)!$ or to use an example $7!$ is equal to $7 \cdot 6!$. Using this fact we can rearrange $(n+1)!$ to be $(n+1) \cdot n!$ so that we can cancel factorials.

$$\rho = \lim_{n\longrightarrow\infty}\left|\frac{((n+1)!)^3}{(3n+4)!}\cdot\frac{(3n+1)!}{(n!)^3}\right| = \lim_{n\longrightarrow\infty}\frac{(n+1)^3(n!)^3}{(3n+4)(3n+3)(3n+2)(3n+1)!}\cdot\frac{(3n+1)!}{(n!)^3} \tag{5}$$

Notice that because each term is positive we can simply drop the absolutely values. Now that we've separated out the factorials we can now cancel like terms.

$$\rho = \lim_{n\longrightarrow\infty}\frac{(n+1)^3}{(3n+4)(3n+3)(3n+2)} \tag{6}$$

Next we want to determine the convergence of the limit. Observe that if we evaluate the limit we will get an indeterminate form so would need to use L'Hopital's rule. However, if we do so we will need to use it 3 times. Rather than doing so we will divide by the highest power of the fraction. To find the highest power in the expression we will do the following:

$$\rho = \lim_{n\longrightarrow\infty}\frac{n^3(1+\frac{1}{n})^3}{n(3+\frac{4}{n})n(3+\frac{3}{n})n(3+\frac{2}{n})} = \lim_{n\longrightarrow\infty}\frac{n^3(1+\frac{1}{n})^3}{n^3(3+\frac{4}{n})(3+\frac{3}{n})(3+\frac{2}{n})} \tag{7}$$

Now we can cancel like terms and evaluate the limit as n approaches infinity

$$\rho = \lim_{n\longrightarrow\infty}\frac{(1+\frac{1}{n})^3}{(3+\frac{4}{n})(3+\frac{3}{n})(3+\frac{2}{n})} = \frac{1}{27} \tag{8}$$

Since we just found that $\rho = 1/27$ which is clearly less than one, then we can say that our series in equation (3) converges.

Working with infinite series is consistently where students struggle the most so please do not hesitate for help, myself or another tutor will be glad to help. If you would like to watch a short video that works a couple examples using ratio test please click **HERE**. [2]

1.2 Root Test

The second major test of convergence we review in this resource is root test. With a few important differences, root test will look very similar to ratio test. Suppose we have the series:

$$\sum_{n=0}^{\infty} a_n \tag{9}$$

We determine its convergence by using the root test; also commonly referred to as the *nth*-root test. To determine its convergence or divergence we need to know the following limit exists:

$$L = \lim_{n\longrightarrow\infty}\sqrt[n]{|a_n|} = \lim_{n\longrightarrow\infty}(a_n)^{1/n} \tag{10}$$

It is crucial that we take the absolute value of a_n otherwise we might get an answer as a complex number which beyond the scope of this course. If we have find that $L < 1$ then $\sum a_n$ converges absolutely. Notice that because we take the absolute value of a_n we are able to conclude the series converges absolutely. However, if we find that $L > 1$ then $\sum a_n$ diverges. Finally, if we know that $L = 1$ then the test is inconclusive. Notice how we able to draw the same conclusions as the ratio test but by taking a different limit. We will see that both of these test are extremely useful when talking about Power Series and Taylor Series.

Let's work an example. Suppose we are asked to determine if the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{e^n}{n^n} \tag{11}$$

We can see that series that look similar to this are perfect candidates for using root test. To determine the convergence we need to set up root test and then evaluate the limit.

$$L = \lim_{n \longrightarrow \infty} \sqrt[n]{\left| \frac{e^n}{n^n} \right|} = \lim_{n \longrightarrow \infty} \left(\left| \frac{e^n}{n^n} \right| \right)^{1/n} \tag{12}$$

The next few steps we need to make sure that remember exponential algebra rules. Additionally we can see that all of our terms are positive so we can drop the absolute value without making any changes.

$$L = \lim_{n \longrightarrow \infty} \left(\left| \frac{e^n}{n^n} \right| \right)^{1/n} = \lim_{n \longrightarrow \infty} \left(\frac{e^n}{n^n} \right)^{1/n} = \lim_{n \longrightarrow \infty} \frac{e^{n/n}}{n^{n/n}} \tag{13}$$

After some simplification of the exponents our next step is to evaluate the limit as n approaches infinity.

$$L = \lim_{n \longrightarrow \infty} \frac{e^{n/n}}{n^{n/n}} = \lim_{n \longrightarrow \infty} \frac{e}{n} = \frac{e}{\infty} = 0 \tag{14}$$

Notice that we found $L = 0$ and so our test is inconclusive. Another viable option would be to consider using geometric series to determine convergence. Notice that since $e \approx 2.71828$ it follows that for $n \geq 3$ we have $(e/n) < 1$. However, since our original series goes from 1 to infinity we need to re-index so that we can apply geometric series. Therefore we can perform the following:

$$\sum_{n=1}^{\infty} \frac{e^n}{n^n} = e + \frac{e^2}{2} + \sum_{n=3}^{\infty} \frac{e^n}{n^n} \tag{15}$$

Finally we are able to conclude that our original series converges and if needed we can find where our series converges to.

If you need more help working with series please schedule a 1-on-1 appointment with myself or another tutor. To watch a video that works another example of root test please click **HERE** [3]. If you have time I also recommend watching this slightly longer video that works multiple examples of root test which you find by clicking **HERE** [4].

References

[1] J. Rogawski, C. Adams, and R. Franzosa, *Calculus: Early Transcendentals*, 4th ed. New York: W. H. Freeman, Dec. 2018.

[2] patrickJMT, "Using the Ratio Test to Determine if a Series Converges #1," Apr. 2008. [Online]. Available: https://www.youtube.com/watch?v=iy8mhbZTY7g

[3] patrickJMT, "The Root Test - Another Example, #1," Jul. 2011. [Online]. Available: https://www.youtube.com/watch?v=nTfJCbqHyxI

[4] The Organic Chemistry Tutor, "Root Test," Mar. 2018. [Online]. Available: https://www.youtube.com/watch?v=ahf0eXOll1M