MTH 1322: Calculus II
Week 12 Tutoring Resources

Ethan Reyes

April 5th, 2021

Welcome Calculus II tutors and students! In this week's resource we will be continuing our work with infinite series. We will go over new ways to determine convergence for different types of series. For more help with these topics please schedule a 1-on-1 visit with me or another tutor. **Please visit baylor.edu/tutoring to make an appointment and to reserve a spot for the Calculus II group tutoring session every Tuesday at 6:30pm.** If you would like to view any of the previous resources please click **HERE**.

Overview[1]

KEYWORDS: Limit and Direct Comparison Test / Alternating Series / Absolute and Conditional Convergence

1 New Topics

1.1 Comparison Tests: Limit / Direct Comparison Test

Recall that when we worked with improper integrals we had a direct comparison that we used to compare complex looking integrals to a more simplified version which allowed us to draw conclusions about convergence. As we will see, the direct comparison test will look very similar to the direct comparison test for improper integrals. Beyond the direct comparison, we will also need to learn about the limit comparison test which will prove to be very useful.

Let's start with the direct comparison test. If we know there exists an $M > 0$ such that $0 \leq a_n \leq b_n$ for all $n \geq M$ then if:

$$\sum b_n \quad \text{converges, then} \quad \sum a_n \quad \text{also converges} \tag{1}$$

On the other hand if we have $0 \leq b_n \leq a_n$

$$\sum b_n \quad \text{diverges, then} \quad \sum a_n \quad \text{also diverges} \tag{2}$$

If we were to look at the direct comparison test for improper integrals we would find that the only difference is that we are using integrals and not infinite summations. Let's work an example: suppose we are asked to determine if the following series converges.

$$\sum_{n=1}^{\infty} \frac{1}{n^n} \tag{3}$$

Looking closely at this series we can see that it is almost in the form of a geometric series. If we examine the first few terms in the sequence we can see the following:

$$a_n = \frac{1}{n^n} : \ 1, \frac{1}{2^2}, \frac{1}{3^3}, \frac{1}{4^4}, \dots \tag{4}$$

[1]The information used to create this resource was taken from this source: [1]

Notice that each term in the sequence is less than 1 which is one of the requirements to use geometric series test. This tells us that it might useful if we can find a geometric series that is greater than our a_n. Notice that for $n \geq 2$ the sequence $(\frac{1}{2})^n$ is greater than the sequence $(\frac{1}{n})^n$. But since we know the infinite series for $(\frac{1}{2})^n$ is geometric with $|r| < 1$ the series converges. Therefore:

$$\sum_{n=1}^{\infty} \frac{1}{n^n} \leq \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \tag{5}$$

Thus our original series converges by direct comparison to another convergent series.

Another comparison test that is often very useful is the limit comparison test. To use the limit comparison test we need to make sure that we have the following:

$$L = \lim_{n \longrightarrow \infty} \frac{a_n}{b_n} \tag{6}$$

where a_n and b_n are both positive. Note that we do not need to know whether $a_n \leq b_n$ to make the following conclusions. If we know that $L > 0$ then $\sum a_n$ converges if and only if $\sum b_n$ converges. This implies the contrary is true also, that $\sum a_n$ diverges if and only if $\sum b_n$ diverges as well. We also know that if $L = \infty$ and $\sum a_n$ converges, then $\sum b_n$ will also converge. Lastly we know that if $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ will also converge. We will soon see that then limit comparison is very useful when working with series.

Let's work an example. Suppose we are asked to prove the following series converges:

$$\sum_{n=0}^{\infty} \frac{n^2}{n^4 - n - 1} \tag{7}$$

Our first step is to determine what our a_n and b_n will be. We typically want to let our original series be our a_n so all that is left is pick a b_n. If we were to look at the highest powers in the numerator and the highest power in the denominator respectively we would end up with this fraction: $\frac{n^2}{n^3} = \frac{1}{n}$. Therefore we want to let our $b_n = \frac{1}{n}$. Now our step is to apply our limit comparison test:

$$\lim_{n \longrightarrow \infty} \frac{\frac{n^2}{n^4 - n - 1}}{\frac{1}{n^2}} = \lim_{n \longrightarrow \infty} \frac{n^2}{n^4 - n - 1} \cdot \frac{n^2}{1} \tag{8}$$

Notice now that we can simplify further then divide by the highest power so that we apply our limit.

$$\lim_{n \longrightarrow \infty} \frac{n^4}{n^4 - n - 1} = \lim_{n \longrightarrow \infty} \frac{1}{1 - 1/n^3 - 1/n^4} = 1 = L \tag{9}$$

Since we now know that $L = 1 > 0$ and we know that the series of $\sum b_n = \sum \frac{1}{n^2}$ converges, it follows that our original series converges as well by the limit comparison test.

Working with infinite series is consistently where students struggle the most so please do not hesitate for help, myself or another tutor will be glad to help. If you would like to watch a video that proves the limit comparison test please click **HERE** [2]. To watch another example of the limit comparison test please click **HERE** [3].

1.2 Alternating Series: Absolute / Condition Convergence

The next type of series will work with are alternating series. Until this point we have been working with positive series but now we will consider how to work with series that alternate between positive and negative terms. The following series is an example of an alternating series:

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \tag{10}$$

Notice that if look at the even and odd terms of the sequence, $(-1)^n \frac{1}{n}$, the terms alternate between positive and negative since we have $(-1)^n$ as part of the sequence. Determining the convergence of these types of series is rather similar to determining convergence of positive series with a few extra stipulations. For an alternating series $\sum a_n$, we say the series converges absolutely if the series $\sum |a_n|$ converges. Note that this also implies $\sum a_n$ also converges. After we take the absolute value of the series we can use any of our current methods to determine convergence of $\sum |a_n|$. Additionally, we say the series $\sum a_n$ converges conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges. To determine if $\sum a_n$ converges we can apply the alternating series test. The alternating series test states that if we have $\sum(-1)^n b_n$, for some positive decreasing sequence b_n, then if:

$$\lim_{n \longrightarrow \infty} b_n = 0 \tag{11}$$

then the series $\sum(-1)^n b_n$ converges. Let's work some examples to help clarify what we mean by each of these statements.

Suppose we are asked to determine if the following series converges absolutely, conditionally, or not at all:

$$\sum_{n=1}^{\infty} \frac{(-1)^n e^{-n}}{n^2} \tag{12}$$

If we separate the $(-1)^n$ from the series to consider the sequence e^{-n}/n^2 we have a positive decreasing sequence and so we can apply the Alternating Series test. Therefore, let's examine the following limit

$$\lim_{n \longrightarrow \infty} \frac{e^{-n}}{n^2} = \lim_{n \longrightarrow \infty} \frac{1}{e^n n^2} = \frac{1}{\infty} = 0 \tag{13}$$

Now that we know the series converges. We need to determine if it converges absolutely or conditionally. So now we consider the following

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n e^{-n}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{e^n n^2} \tag{14}$$

Now we are free to use any of the tests that we know. Looking at this series it seems that we might be able to use a direct comparison to a p-series series. So consider the following inequality: $n^2 \leq e^n n^2$ for all n. Now if we divide both sides by $(e^n n^2)(n^2)$ we would have the following inequality:

$$\frac{1}{e^n n^2} \leq \frac{1}{n^2} \tag{15}$$

Since we know that the infinite series $1/n^2$ converges by p-series test, we know that the series $1/(e^n n^2)$ also converges. Thus we find our original series converges absolutely.

If you would like to watch a video that works example problems of both conditional and absolute convergence please click **HERE** [4].

References

[1] J. Rogawski, C. Adams, and R. Franzosa, *Calculus: Early Transcendentals*, 4th ed. New York: W. H. Freeman, Dec. 2018.

[2] "The Limit Comparison Test - Proof." [Online]. Available: https://www.youtube.com/watch?v=d6EBbmJVTqk

[3] "Limit Comparison Test." [Online]. Available: https://www.youtube.com/watch?v=LBxYQ0TJxYM

[4] The Organic Chemistry Tutor, "Absolute Convergence, Conditional Convergence, and Divergence," Mar. 2018. [Online]. Available: https://www.youtube.com/watch?v=FPK6LO1iiXc