MTH 1322: Calculus II
Week 10 Tutoring Resources

Ethan Reyes

March 22, 2021

Welcome Calculus II tutors and students! In this week's resource we will be working with infinite series and reviewing how to determine convergence of these series. For more help with these topics please schedule a 1-on-1 visit with me or another tutor. **Please visit baylor.edu/tutoring to make an appointment and to reserve a spot for the Calculus II group tutoring session every Tuesday at 6:30pm.** If you would like to view any of the previous resources please click **HERE**.

Overview[1]

KEYWORDS: Geomtric Series / Integral test / Limit test

1 New Topics

1.1 Summing an Infinite Series

In the last resource we worked with sequences. For this next topic we will discuss what to do when summing those sequences. Furthermore, we will also discuss how to determine convergence of infinitely summed sequences. Recall that sequences are of the form:

$$a_n = \frac{1}{n} \tag{1}$$

or some variation where each term is dependent on n. Before starting infinite series, lets take a look at what happens if we take partial sums. To take these partial sum simply evaluate the sequence by the specified number of terms. For example if we wish to take the partial sum of the sequence in (1) we first evaluate some of the terms in the sequence.

$$a_3 = \frac{1}{1},\ \frac{1}{2},\ \frac{1}{3} \tag{2}$$

Next we want to consider what each partial will be. To do this we add each term together. Notice how the a partial sum where $n = 2$ means we add the first two terms of the sequence together. We see to find a partial sum of $n = 3$ we would add the first three terms together and so on and so forth.

$$S_1 = a_1 \quad S_2 = a_1 + a_2 \Longrightarrow S_n = a_1 + a_2 + a_3 \cdots + a_n \tag{3}$$

For an example we will compute a couple of the partial sums of (1). It is important to note that in general every partial sum will be finite. This means that even the nth partial sum of our sequence will be a finite real number.

$$S_1 = 1 \ \longrightarrow S_2 = 1 + \frac{1}{2} \ \longrightarrow S_3 = 1 + \frac{1}{2} + \frac{1}{3} \tag{4}$$

[1]The information used to create this resource was taken from this source: [1]

To make the leap from these finite partial sums we start by considering the limit of the partial sums. If we let n go to infinity then we formalize the idea as the following:

$$\sum_{n=0}^{\infty} a_n \tag{5}$$

If we find that as n tends towards infinity and our partial sums tend toward a finite number say M, then we say the following limit is true:

$$\lim_{n \longrightarrow \infty} S_n = M \tag{6}$$

If we find that the partial sums are unbounded as n goes to infinity then we can the infinite series (infinite sum) will also be unbounded and thus diverge. Similarly, if are able to determine the convergence of a limit of a sequence, say b_n, then we will able to make inferences about an infinite series regarding b_n. We formalize this idea below and call it the divergence test.

$$\boxed{\lim_{n \longrightarrow \infty} b_n \neq 0 \implies \sum_{n=1}^{\infty} b_n \text{ Diverges}} \tag{7}$$

It important to understand that finding the limit *is* equal to 0 will tell us nothing about the series. As an example if we consider the sequence in equation (1), we know the limit as n goes to infinity is equal to 0 but we will soon find that the infinite series will actually diverge. Let's work an example. Suppose we are asked to determine if the following series converges or diverges:

$$\sum_{k=0}^{\infty} \frac{3k^2+5k-7}{k^2-7} \tag{8}$$

Even though we currently only have one method to test convergence, I highly recommend that you always first consider using limit test when asked to determine if a series converges or diverges. The reason is because it is rather simple compared to the future methods that we will soon learn. To use the limit test we will take the limit as k goes to infinity of the fraction.

$$\lim_{k \longrightarrow \infty} \frac{3k^2+5k-7}{k^2-7} \tag{9}$$

Notice that we will end up with an indeterminate form and so we can apply L'Hopital's rule. Notice that we will need to apply L'Hopital's rule twice because the exponent in the numerator and denominator are both 2.

$$\lim_{k \longrightarrow \infty} \frac{3k^2+5k-7}{k^2-7} = \lim_{k \longrightarrow \infty} \frac{6k+5}{2k} = \lim_{k \longrightarrow \infty} \frac{6}{2} = 3 \tag{10}$$

Since we found that our limit does not equal 0 we know the series will diverge.

Another important concept in this section is the idea of geometric series. Geometric series are important because they are one of the few infinite series that we can actually determine the limit of convergence. For most infinite series we will only be able to determine divergence or convergence. If we have that $c \neq 0$ and $|r| < 1$ then we know the following about the infinite series:

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + \cdots = \frac{c}{1-r} \tag{11}$$

It is also important to know that if $|r| \geq 1$ then the series diverges. As an example Suppose we are asked to determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{8+2^n}{5^n} \tag{12}$$

Looking at this series we can see that it looks relatively similar to the form we see in equation (11). Because of this it reasonable to attempt to convert our series so that we can apply equation (11) and determine convergence. Using properties of series we can do the following:

$$\sum_{n=0}^{\infty} \frac{8}{5^n} + \frac{2^n}{5^n} = 8\sum_{n=0}^{\infty} \frac{1}{5^n} + \sum_{n=0}^{\infty} \frac{2^n}{5^n} = 8\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n \tag{13}$$

Notice that now we have two infinite series in the general form so therefore we can easily determine if the series converge or diverge. For the first series observe that $|r| = \frac{1}{5} < 1$ which by definition converges. Similarly we find the second series since $|r| = \frac{2}{5} < 1$ it also converges. Since the series converges and because of the special properties of geometric series we can calculate the value to which the series converges to. Using the formula from equation (11) we find:

$$\sum_{n=0}^{\infty} \frac{8+2^n}{5n} = \frac{8}{1-\frac{1}{5}} + \frac{1}{1-\frac{2}{5}} = 10 + \frac{5}{3} \tag{14}$$

If you need more help working with series please schedule a 1-on-1 appointment with myself or another tutor. If you would like to watch a short video that discusses the divergence test you can do so by clicking **HERE** [2]. If would like to watch a proof explaining geometric series and why it converges please click **HERE** [3].

1.2 Convergence of Series with Positive Terms

Moving forward we will begin more tests for convergence. The first new test we will discuss is the integral test. If we can find a function $f(x)$ such that we have $a_n = f(n)$ then we will able to make inferences about the infinite series about a_n. The only stipulation is that we must have f as a decreasing positive function. Most of time we will able to find $f(x)$ by observation. For example, consider the sequence: $a_n = \frac{1}{n}$. Notice that $f(x) = \frac{1}{x}$ will yield the exact same numbers for integer values of x. Since we found a positive decreasing function relating to a_n we now apply the definition.

$$\text{If } \int_1^{\infty} f(x)\,dx \;\text{ converges, then } \sum_{n=1}^{\infty} a_n \text{ converges} \tag{15}$$

Notice that we return back to improper integrals. Therefore it is imperative that we understand how to work with improper integrals so we can apply them to infinite series.

$$\text{If } \int_1^{\infty} f(x)\,dx \;\text{ diverges, then } \sum_{n=1}^{\infty} a_n \text{ diverges} \tag{16}$$

In the next concept we will see direct applications of the integral test and the results it yields.

P-series are a special case of infinite series where the integral test will allow us to determine convergence of series in a special form. In this section we will also provide an explanation to why the infinite sum of the sequence $\frac{1}{n}$ diverges. P-series of the form:

$$p - \text{series:} \quad \sum_{n=1}^{\infty} \frac{1}{n^p} \tag{17}$$

Here we let p be any real number. Notice that p can be equal to any real number and is not dependent on the series itself ie p does not change with n. As we mentioned earlier since for all real numbers p related to $1/n^p$ we can find a positive decreasing function $f(x) = 1/x^p$ we can apply integral test.

$$p > 1: \int_1^{\infty} \frac{1}{x^p} = L \Longrightarrow \text{convergent} \tag{18}$$

Again we see the importance of being able to work with improper integrals.Observe that we have reduced the infinite series to a p-integral. We know there are different cases of p-integrals that will help us determine convergence of the infinite series.

$$p \leq 1: \int_1^\infty \frac{1}{x^p} = \infty \Longrightarrow \text{divergent} \tag{19}$$

Thus we finally formally define why the infinite sum of $\frac{1}{n}$ diverges. We can see that by properties of improper integrals values of $p \leq 1$ will diverge and thus the corresponding infinite series will also diverge.

Working with infinite series is consistently where students struggle the most so please do not hesitate for help, myself or another tutor will be glad to help. If you would like to watch a short video working with integral test please click **HERE** [4].

References

[1] J. Rogawski, C. Adams, and R. Franzosa, *Calculus: Early Transcendentals*, 4th ed. New York: W. H. Freeman, Dec. 2018.

[2] "Test for Divergence for Series, Two Examples," Jul. 2011. [Online]. Available: https://www.youtube.com/watch?v=DcduHCJuiWk

[3] "Geometric Series - Proof of the Sum of the first n terms : ExamSolutions." [Online]. Available: https://www.youtube.com/watch?v=xfzcUkcLXXA

[4] "Integral test | Series | AP Calculus BC | Khan Academy," Aug. 2014. [Online]. Available: https://www.youtube.com/watch?v=u1UKIljUWuc