MTH 1322: Calculus II
Week 7 Tutoring Resources

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Welcome Calculus II tutors and students! This week's resource we will cover improper integrals again and using integrals to find probability. For more help with these topics please schedule a 1-on-1 visit with me or another tutor. **Please visit baylor.edu/tutoring to make an appointment and to reserve a spot for the Calculus II group tutoring session every Tuesday at 6:30pm.** If you would like to view any of the previous resources please click **HERE**.

Overview [1]

1.1 Improper Integrals
1.2 Probability Using Integrals
2 References

KEYWORDS: Improper Integrals (comparison test) / Probability Using Integrals

1 New Topics

1.1 Improper Integrals (cont'd)

Since we did not cover all of improper integrals in the last resource we will continue to cover them in this one as well. In last week's resource we discussed some general forms of improper integrals and how to evaluate them. Specifically, we discussed working p integrals. For example recall that for $p > 1$:

$$\int_a^\infty \frac{dx}{x^p} \text{ converges } \quad \int_0^a \frac{dx}{x^p} \text{ diverges} \tag{1}$$

We verified this in last week's resource by working. If you are still hesitant I recommend watching the linked videos in the resource to help clarify. Similarly we also know that for $p < 1$:

$$\int_a^\infty \frac{dx}{x^p} \text{ diverges } \quad \int_0^a \frac{dx}{x^p} \text{ converges} \tag{2}$$

It is also important to know for $p = 1$ both integrals will diverge. In this resource we will apply these concepts so that we make conclusions about more complex looking integrals.

The next important concept from this section is the comparison test. As the name suggest we will be making conclusions about more complex looking improper integrals by comparing them to simpler ones we know more about. Suppose we have two functions $f(x)$ and $g(x)$ and suppose we know that $f(x) \geq g(x) \geq 0$, then:

$$\text{If } \int_a^\infty f(x)\,dx \text{ converges, then } \int_a^\infty g(x)\,dx \text{ also converges} \tag{3}$$

[1] The information used to create this resource was taken from this textbook: [1]

Similarly, we also have that

$$\text{If } \int_a^\infty g(x)\,dx \text{ diverges, then } \int_a^\infty f(x)\,dx \text{ also diverges} \tag{4}$$

Let's work a concrete example to help solidify this concept. Suppose we are asked to determine if the following integral diverges or converges:

$$\int_1^\infty \frac{dx}{x^3+4} \tag{5}$$

One approach would be to recognize that $\frac{1}{x^3+4} < \frac{1}{x^3}$ by looking at the denominator. From there we could make a conclusion about our original integral. However, if this is not immediately obvious i recommend taking a different approach. For now lets consider x^3+4. By observation we can say $x^3+4 > x^3$, therefore if we took the inverse of both functions we would end up with $\frac{1}{x^3} > \frac{1}{x^3+4}$. Therefore we can let $f(x) = \frac{1}{x^3}$ and $g(x) = \frac{1}{x^3+4}$. So now we can apply equation (1) to evaluate this integral:

$$\int_1^\infty \frac{dx}{x^3+4} \leq \int_1^\infty \frac{dx}{x^3} \text{ which converges by equation 1} \tag{6}$$

The second approach is more applicable to different cases, the goal of the thought process is to find a function that we know more about and then use the fact that it converges or diverges to make the same conclusion about the presented function. Improper integrals require lots of practice to fully understand and apply ideas, that being said I recommend watching a short video to help refresh your memory. To watch the video please click **HERE** [2].

1.2 Arc Length and Surface Area

Arc length is another application of integrals. To say we are finding an arc length is equivalent to saying we are finding the length of a curve. After using some theorems we find the general formula for arc length, s, is the following integral [1]:

$$s = \int_a^b \sqrt{1+f'(x)^2}\,dx \tag{7}$$

However, this general formula is only true if $f'(x)$ exists and is continuous on the interval $[a, b]$.

In previous resources we learned how to find volumes of revolution. Now we need to understand how to find the area of a surface of revolution. To find the surface area we first assume that $f(x) \geq 0$ and that $f'(x)$ exists and continuous on the interval $[a, b]$. Given these two conditions we find the surface area, S, of the surface obtained by rotating $f(x)$ about the x-axis for $a \leq x \leq b$ is equal to the following integral [1]:

$$S = 2\pi \int_a^b f(x)\sqrt{1+f'(x)^2}\,dx \tag{8}$$

Let's work an example to help explain this concept. Suppose we are asked to calculate the surface area of a sphere of radius R. Recall that the formula for a circle is $y^2 + x^2 = R^2$, where R is the radius. So solving for y we can see that $y = f(x) = \sqrt{R^2 - x^2}$. Now we need to take the derivative: $f'(x) = -\frac{x}{\sqrt{R^2-x^2}}$. Therefore $1 + f'(x) = 1 + \frac{x^2}{R^2-x^2} = \frac{R^2}{R^2-x^2}$.
Now putting it all together in our integral:

$$S = 2\pi \int_{-R}^{R} \sqrt{R^2-x^2}\sqrt{\frac{R^2}{R^2-x^2}}\,dx \;=\; 2\pi \int_{-R}^{R} \sqrt{R^2-x^2}\frac{R^2}{\sqrt{R^2-x^2}}\,dx \tag{9}$$

$$= 2\pi R \int_{-R}^{R} dx \;=\; 2\pi R(2R) \;=\; 4\pi R^2 \tag{10}$$

As you can see just derived the formula for volume of any sphere with radius R which we were told to memorize back in geometry. Finding surface area is very similar to what we did when learning volumes of revolution. To watch a short video working a different example please click **HERE** [3].

References

[1] J. Rogawski, C. Adams, and R. Franzosa, *Calculus: Early Transcendentals*, 4th ed. New York: W. H. Freeman, Dec. 2018.

[2] "Introduction to improper integrals (video)." [Online]. Available: https://www.khanacademy.org/math/ap-calculus-bc/bc-integration-new/bc-6-13/v/introduction-to-improper-integrals

[3] "Worked example: arc length | Applications of definite integrals | AP Calculus BC | Khan Academy." [Online]. Available: https://www.youtube.com/watch?v=OhISsmqv4_8