MTH 1322: Calculus II
Week 5 Tutoring Resources

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Welcome Calculus II tutors and students! Last week was exam week for a couple of the sections so the material covered in this week's resource will be comparatively short. **Please visit baylor.edu/tutoring to make an appointment and to reserve a spot for the Calculus II group tutoring session every Tuesday at 6:30pm.**

Overview[1]

KEYWORDS: Method of partial fractions (polynomial long division) / L'Hopital's Rule

1 New Topics

1.1 Method of Partial Fractions cont'd

Recall that we use partial fractions to integrate $f(x)$ when $f(x) = \frac{P(x)}{Q(x)}$. Where $P(x)$ and $Q(x)$ are both polynomials. To use this method we must first make sure that the degree of P is less than the degree of Q [1]. If the degree of P is greater than Q, then we must use polynomial long division to reduce the degree of P. Let's work an example; suppose we are asked to evaluate the following integral:

$$\int \frac{x^5 + 2x^4 + x^3 - x^2 - 22x + 15}{x^2 + 2x - 3}\, dx \tag{1}$$

We can see that the degree of P is equal 5 whereas the degree of Q is equal to 2. So before can apply partial fractions we must perform polynomial long division of the two polynomials:

$$\begin{array}{r|rrrrrr} & x^3 & & & +4x & -9 & \\ \hline x^2+2x-3 & x^5+2x^4 & +x^3 & -x^2 & -22x & +15 \\ & -x^5-2x^4 & +3x^3 & & & \\ \hline & & 4x^3 & -x^2 & -22x & \\ & & -4x^3 & -8x^2 & +12x & \\ \hline & & & -9x^2 & -10x & +15 \\ & & & 9x^2 & +18x & -27 \\ \hline & & & & 8x & -12 \end{array} \tag{2}$$

We can see that we have a remainder of $8x - 12$, so we take our remainder and leave that on top of the fraction, while taking our quotient out of the fraction; resulting in the following fraction.

$$(x^3 + 4x - 9) + \frac{8x - 12}{x^2 + 2x - 3} \tag{3}$$

[1]The information used to create this resource was taken from this textbook: [1]

Now we can see that P has degree 1, which is less than Q with degree 2. Therefore we can properly apply integration by parts to our new fraction.

$$\int (x^3 + 4x - 9) + \frac{8x - 12}{x^2 + 2x - 3}\, dx \;=\; \int (x^3 + 4x - 9)\, dx + \int \frac{8x - 12}{(x + 3)(x - 1)}\, dx \tag{4}$$

If we ignore the integral on the right fraction we can see that if we apply method of partial fractions we get the following equation:

$$\frac{8x - 12}{(x + 3)(x - 1)} = \frac{A}{x + 3} + \frac{B}{x - 1} \Longrightarrow 8x - 12 = A(x - 1) + B(x + 3) \tag{5}$$

Our goal is to solve for A and B so we want to choose values of x that making solving for them easy. In this case we can see that if we choose $x = 1, -3$ we will be able to easily solve for A and B. For $x = 1$ we have:

$$8 - 12 = A(0) + B(4) \;\longrightarrow\; -4 = 4B \;\longrightarrow\; B = -1 \tag{6}$$

For $x = -3$ we have:

$$-8 - 12 = A(-4) + B(0) \;\longrightarrow -20 = (-4)A \;\longrightarrow A = 5 \tag{7}$$

Therefore we can simplify and rewrite the integral in (4) as the following:

$$\int (x^3 + 4x - 9)\, dx + \int \frac{5}{x + 3} - \frac{1}{x - 1}\, dx \tag{8}$$

Thus, after basic integration we see the final solution is:

$$\frac{1}{4}x^4 + 2x^2 + 9x + 5\ln|x + 3| - \ln|x - 1| + C \tag{9}$$

Method of partial fractions has a plethora of different cases to consider which is why I recommend watching a video or two on the topic. The first video is a short introduction to the concept which can found by clicking **HERE** [2]. If you have time I also recommend watching this longer video that works more example problems involving partial fractions (**HERE**) [3].

1.2 L'Hoptial's Rule

Since there were a few sections that covered L'Hopital's Rule for the first time last week it seems fit to review another example. Recall that L'Hopital's Rule is defined to be:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \tag{10}$$

where a can be $\pm\infty$ or any real number. But we can only use L'Hopital's rule when we find that taking the limit of $\frac{f(x)}{g(x)}$ yields an indeterminate form. Some examples of indeterminate forms: $\frac{0}{0}$; $(0)(\pm\infty)$; 1^∞ ; 0^0 ; ∞^0; $\infty - \infty$; $\frac{\infty}{\infty}$. Now suppose we have the following limit:

$$\lim_{x\longrightarrow 0} \ln x \sin x \tag{11}$$

Notice that if we evaluate the limit we get indeterminate form: $0 \cdot (\infty)$. Therefore, we can apply L'Hopital's rule. Before we do so it would be beneficial to rewrite our original equation as the following:

$$\lim_{x\longrightarrow 0} \frac{\ln x}{\frac{1}{\sin x}} = \lim_{x\longrightarrow 0} \frac{\ln x}{\csc x} \tag{12}$$

Now if we apply L'Hopital's rule to our new equation we end up with the following:

$$\lim_{x\longrightarrow 0} \frac{\frac{1}{x}}{-\csc x \cot x} \tag{13}$$

Now we can apply trigonometric identities to simplify the fraction. We will use the fact that $\csc x = \frac{1}{\sin x}$ and $\cot x = \frac{\sin x}{\cos x}$. Thus our new limit becomes:

$$\lim_{x\longrightarrow 0} \frac{\frac{1}{x}}{-\csc x \cot x} = \lim_{x\longrightarrow 0} \frac{\sin^2 x}{x \cos x} \tag{14}$$

For our next step we will split the fraction so that we can apply our limit separately.

$$\lim_{x\longrightarrow 0} \frac{1}{\cos x} \frac{\sin x}{x} \sin x \tag{15}$$

Going back to calculus I, we know that the limit as x approaches 0 for the fraction: $\frac{\sin x}{x} = 1$ is an identity we need to know. Therefore applying limits we have the following:

$$\lim_{x\longrightarrow 0} \frac{1}{\cos x} \frac{\sin x}{x} \sin x = \frac{-1}{1} \cdot 1 \cdot 0 = 0 \tag{16}$$

So by using L'Hoptial's rule we found:

$$\lim_{x\longrightarrow 0} \sin x \ln x = 0 \tag{17}$$

Next week we will be covering improper integrals so it is a good idea to review limits in general. That being said, consider the following limit:

$$y = \lim_{x\longrightarrow \infty} x^{1/\ln x} \tag{18}$$

x evaluating the limit we see that we have $\infty^{\frac{1}{\infty}} = (\infty)^0$, but before we apply L'Hopital's Rule it is in our interest to take the natural log of both sides, giving us:

$$\ln(y) = \ln(\lim_{x\longrightarrow \infty} x^{1/\ln x}) = \frac{1}{\ln x} \ln(\lim_{x\longrightarrow \infty} x) \tag{19}$$

By properties of limits we can pull limit out to be in front of $\frac{1}{\ln x} \ln x$. Notice that this now gives us the following equation:

$$\ln y = \lim_{x\longrightarrow \infty} (1) \tag{20}$$

Therefore since the limit of 1 as x goes to infinity is 1, we can drop the limit and we would then have: $\ln y = 1$. Therefore we need to take the exponential of both sides so our answer is of the form $y = e^1$. Thus our final answer is:

$$y = \lim_{x\longrightarrow \infty} x^{1/\ln x} = e \tag{21}$$

Remember to always think outside the box when you come across a problem that does not appear to have a solution. Some problems might require you to use trig-identities or as in the example above, to use properties of the natural log. To watch a short video click **HERE** [4].

2 Topics Previously Covered

2.1 Weeks 1 & 2

In weeks 1 & 2 we reviewed part of calculus I, specifically U-Substitution. We also covered some new topics, such as Trigonometric Substitution and Trigonometric integrals. We worked some examples with integration by parts. If you would like to view this Resource please click **HERE**.

2.2 Week 3

In week 3 we discussed how to use integrals to find average value, density, and volumes. We also worked examples with Volumes of revolution for disk, washer, and shell method. Additionally we worked an example of solving integrals using method of partial fractions. If you would like to view this resource please click **HERE**.

References

[1] J. Rogawski, C. Adams, and R. Franzosa, *Calculus: Early Transcendentals*, 4th ed. New York: W. H. Freeman, Dec. 2018.

[2] "Integration with partial fractions (practice)." [Online]. Available: https://www.khanacademy.org/math/ap-calculus-bc/bc-integration-new/bc-6-12/e/integration-of-rational-functions-by-division-and-partial-fractions

[3] "Integration By Partial Fractions." [Online]. Available: https://www.youtube.com/watch?v=r5MIraVUVUA&t=982s

[4] K. Academy, "Introduction to l'hopital's rule," Youtube, 2010. [Online]. Available: https://www.youtube.com/watch?v=PdSzruR5OeE