MTH 2311 Linear Algebra
Week 12 Resources

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Major Topics:

1. Orthogonality and Projections
2. Orthogonal Subspaces

Textbook Material:

Linear Algebra and Its Applications, 5th Edition by Lay and McDonald
Sections 6.2-6.3

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1 Conceptual Review

Author's Remark

If you are reading this, then it means you have made it to the final weeks of your Linear Algebra course! I hope that this course was both challenging and enthralling for you. As we are moving into the final chapters of the Syllabus, it is usually a good idea to start generating outlines of all of the major results so far so that you will be prepared for the final exam. Although Linear algebra is mostly a cumulative course, there may be some sections of the material that may be worth revisiting so that you can be sure you have all of your questions answered before final exams start. The end is in sight, so don't give in just yet!

– Colin B.

Orthogonality and Projections

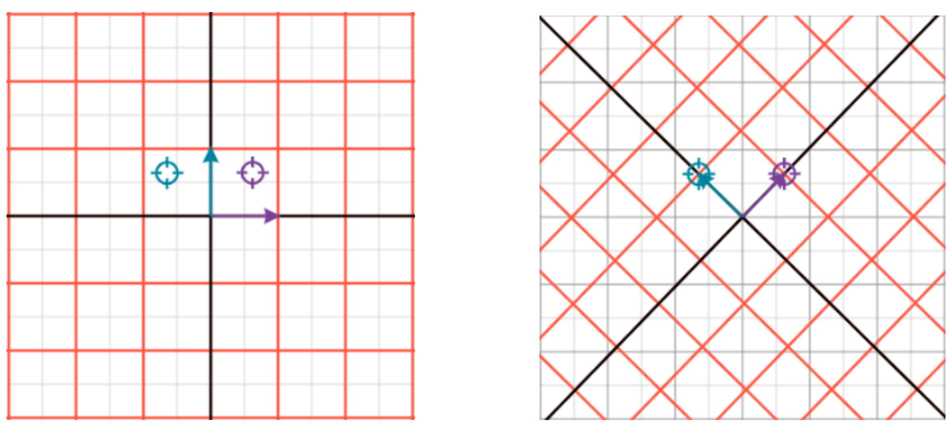
Early in the semester you may have touched on the concept of orthogonality. We say that two vectors are *orthogonal*, if the angle between them is $\pi/2$ radians (90 degrees). From the geometric definition of the dot product, we recall that of two vector $\mathbf{a}$ and $\mathbf{b}$ have the angle θ between them:

$$\mathbf{a} \bullet \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos(\theta)$$

(Recall that $\|\mathbf{x}\|$ denotes the "norm" (length) of $\mathbf{x}$)

Thus, if $\theta = \pi/2$, which means $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, then $\mathbf{a} \bullet \mathbf{b} = 0$. From this, you might recall the familiar result that *two vectors are orthogonal if and only if their dot product is 0.*

From this definition, it follows intuitively that if vectors are orthogonal, then they must be linearly independent; in fact, every vector space can be written as as the span of a set of orthogonal (hence linearly independent) vectors. This is referred to as an *orthogonal basis.* We can even take things a step further and impose that the norm (length) of each vector in the set of basis vectors be 1. This is often referred to as an *orthonormal basis.* One example of an orthonormal basis that you may be familiar with is the natural basis for $\mathbb{R}^n$, which comprise the columns of the $n \times n$ identity matrix $\mathbf{I}_n$. However, orthonormal bases are not unique bases, as we can simply "rotate" the natural basis vectors to obtain a new orthonormal basis. Consider, for example, the natural basis vectors for $\mathbb{R}^2$ rotated counter-clockwise by $\pi/2$ radians:



Observe that both the natural basis vectors and the rotated natural basis vectors provide an orthonormal basis for $\mathbb{R}^2$. We can use this example to illustrate two useful principles of orthonormal bases:

1. Any two orthonormal bases differ only by rotations and reflections.

2. If the columns of an $n \times n$ matrix $\mathbf{A}$ form an orthonormal basis for $\mathbb{R}^n$, then the transformation $\mathbf{A}$ encodes a rotation and series of reflections about the origin; that is, for any vector $\mathbf{x}$, $\|\mathbf{A}\mathbf{x}\| = \|\mathbf{x}\|$.

In short, we can say that matrices with orthonormal columns (often referred to as *orthogonal matrices*) are "norm-preserving" transformations that change the direction of vectors, but not their magnitude.

Orthogonal Subspaces

This notion of orthogonality can also apply to vector subspaces. We say that two vector subspaces $\mathcal{U}$ and $\mathcal{V}$ are *orthogonal subspaces* if every vector $\mathbf{v}$ in $\mathcal{V}$ is orthogonal to every vector $\mathbf{u}$ in $\mathcal{U}$. You may recall earlier that we proved for any matrix $\mathbf{A}$ every vector in the null space of $\mathbf{A}$ is orthogonal to every other vector in the row space of $\mathbf{A}$ (see the Examples section below for a repeat of the proof). By definition, this means that null($\mathbf{A}$) is a subspace orthogonal to range($\mathbf{A}^T$). (This is usually denoted by: null($\mathbf{A}$) $\perp$ range($\mathbf{A}^T$)).

Interestingly, there are other instances of subspace orthogonality that deal with the four fundamental subspaces:

Orthogonality properties of the Four Fundamental Subspaces
Let $\mathbf{A} \in \mathbb{R}^{m \times n}$:

1. $\dim(\text{null}(\mathbf{A})) + \dim(\text{range}(\mathbf{A}^T)) = m$

2. $\text{null}(\mathbf{A}) \perp \text{range}(\mathbf{A}^T)$

3. $\dim(\text{null}(\mathbf{A}^T)) + \dim(\text{range}(\mathbf{A})) = n$

4. $\text{null}(\mathbf{A}^T) \perp \text{range}(\mathbf{A})$

What this tells us, is not just information about the dimension of the four fundamental subspaces, but also how they relate to each other. In particular it shows that the row space (the range of $\mathbf{A}^T$) is perpendicular to the null space and the column space (the range of $\mathbf{A}$) is perpendicular to left null space. In the later chapters of your textbook, you might also run into the following corollary:

$$\text{null}(\mathbf{A}) \oplus \text{range}(\mathbf{A}^T) \equiv \mathbb{R}^m \qquad \text{(from properties 1 and 2)}$$
$$\text{null}(\mathbf{A}^T) \oplus \text{range}(\mathbf{A}) \equiv \mathbb{R}^n \qquad \text{(from properties 3 and 4)}$$

For more info on the $\oplus$ relation, see the FAQ section below.

2 Frequently Asked Conceptual Questions

1. **I have seen in the textbook the notation of the 'direct sum' operator $\oplus$. What does this mean in terms of subspaces?**

The $\oplus$ operator denotes that any two subspaces can be combined such that the span of the vectors from both vector spaces produce the resulting vector space. However, it also requires that the intersection of the two 'summand' spaces (the spaces being combined together) have what is called a *trivial intersection*, meaning that the only vector contained in both of the vector spaces is $\mathbf{0}$, which must be a member of both by definition. One example is below:

Let $U = \text{span}(\{\mathbf{e_1}, \mathbf{e_2}\})$ and $V = \text{span}(\{\mathbf{e_3}\})$ where U, V are subspaces of $\mathbb{R}^3$, and $\mathbf{e_i}$ is the ith natural basis vector (ith column of I_3). Then we can write:

$$U \oplus V \equiv \mathbb{R}^3$$

To summarize, do no think of the $\oplus$ as an operation (at least in the context of this course), but think of it as part of a statement about subspace structure.

3 Examples

N.B: The examples below are more conceptually oriented, because they tend to be the ones that students have difficulty with. For some calculation-oriented examples, see the textbook.

1. **(Repeated from the week 7 resource) Let A be a matrix in $\mathbb{R}^{n \times n}$. Do/Show the following:**

   (a) Every vector in the null space of $\mathbf{A}$ is orthogonal to every vector in the range of $\mathbf{A}^T$ (sometimes called the 'row space' of $\mathbf{A}$).

   (b) Using your answer above, find all vectors in the intersection of the null space of $\mathbf{A}$ and the range of $\mathbf{A}^T$.

   Note: This proof/demonstration will most likely be one of the toughest that you will encounter, so if you have a solid grasp on this, then you should be well-equiped to tackle most other proof-like problems in this course.

   If a vector $\mathbf{y}$ is in the range of $\mathbf{A}^T$, then $\mathbf{y}$ must be some linear combination of the rows of $\mathbf{A}$ (note we are using *rows* not *columns* here, since we are working with $\mathbf{A}^T$, not $\mathbf{A}$). Letting r_i be the ith row of $\mathbf{A}$, we know that if $\mathbf{y}$ is in the range of $\mathbf{A}$, then we can write:

$$\mathbf{y} = \sum_{i=1}^{n} c_i \mathbf{r_i} \text{ for constants } c_i.$$

   Likewise, if $\mathbf{z}$ is in the null space of $\mathbf{A}$, then we can write $\mathbf{Az} = \mathbf{0}$, so for each $\mathbf{r_i}$, $\mathbf{r_i}^T\mathbf{z} = 0$.

To show that every vector in range($\mathbf{A}^T$) is orthogonal to every vector in null($\mathbf{A}$), it suffices to show that for every $\mathbf{y}$ and $\mathbf{z}$ as defined above, $\mathbf{y}^T\mathbf{z} = 0$. We can show this with the following:

$$\mathbf{y}^T\mathbf{z} = \left(\sum_{i=1}^{n} \mathbf{r_i} c_i\right)^T \mathbf{z} = \left(\sum_{i=1}^{n} c_i \mathbf{r_i}^T\right) \mathbf{z} = \sum_{i=1}^{n} c_i \mathbf{r_i}^T \mathbf{z} = \sum_{i=1}^{n} (0) = 0$$

(c) Find all vectors in null($\mathbf{A}$) $\cap$ range($\mathbf{A}^T$).

Because null($\mathbf{A}$) $\perp$ (is orthogonal to) range($\mathbf{A}^T$), we observe that the only vector orthogonal to itself is $\mathbf{0}$, and it must be contained in both subspaces. Thus:

$$\text{null}(\mathbf{A}) \cap \text{range}(\mathbf{A}^T) = \{\mathbf{0}\}.$$

Additional References:

I would highly recommend looking into the following resources:

1. *Linear Algebra and Its Applications, 5th Edition* by Lay and McDonald (ISBN-13: 978-0321982384)

2. 3Blue1Brown *Essence of Linear Algebra Series*: www.3blue1brown.com/essence-of-linear-algebra-page

As we approach the end of the Spring semester, please remember that online tutoring continues to be available through individual and group tutoring sessions. Please visit baylor.edu/tutoring to make an appointment and to reserve a spot for the Linear Algebra group tutoring, which happens online every Monday at 5:15 PM CDT.
