MTH 2311 Linear Algebra
Week 11 Resources

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4/20/20

Major Topics:

1. Diagonalization

Textbook Material:

Linear Algebra and Its Applications, 5th Edition by Lay and McDonald
Sections 4.2-4.4

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1 Conceptual Review

Recall from last week that we introduced the concept of characteristic equations with the following theorem:

The Fundamental Theorem of Algebra:
A polynomial of order n has at most n real roots. If we allow for roots to be complex-valued (have an imaginary component) or to be counted multiple times (such as -1 is a repeated root of $x^2 + 2x + 1$), then the polynomial has *exactly* n roots.

If we allow for complex eigenvalues and repeated eigenvalues, then a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ must have *exactly* n eigenvalues. In this course, we will not be working with complex eigenvalues; however, we will encounter repeated eigenvalues. Recall that we can also have eigenvalues of 0, which are addressed in the FAQ section below (repeated from last week).

This week we will be introducing the idea of *diagonalization*, which is a kind of matrix factorization that is both useful in what it says about a matrix and what it allows us to do. In some textbooks, diagonalization is referred to as the *eigenvalue decomposition* of a matrix, because it decomposes a matrix into the product of three matrices. As we will discuss below, however, this factorization is not always possible. Suppose we are given a square matrix $\mathbf{A}$. If the matrix is *diagonalizable*, we can decompose it as the following square matrix product:

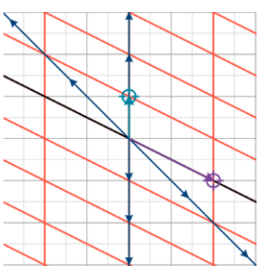
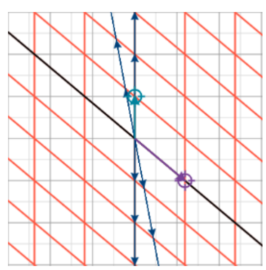
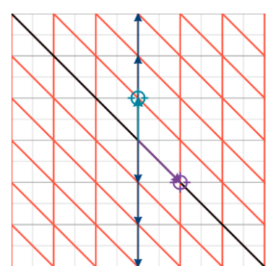
$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, where the following conditions are met:

1. The matrix $\mathbf{D}$ is a diagonal matrix containing the eigenvalues of $\mathbf{A}$. Typically these values are given in order of descending magnitude.

2. The matrix $\mathbf{P}$ consists of the linearly independent eigenvectors of $\mathbf{A}$. Each eigenvector is in the same column of $\mathbf{P}$ as its corresponding eigenvector in d. Typically these vectors are shortened to be of unit length, or are left as convenient integer-valued vectors.

From the definition above, if follows that the square matrix $\mathbf{A}$ is diagonalizable if and only if the dimension of the eigenspaces associated with each distinct eigenvector sum up to be n. If the dimensions of the eigenspaces sum up to be less than n, (following the trend among mathematicians of assigning pejorative names to pathological instances) we call such a matrix *defective*.

One example of a defective matrix is the 2x2 matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$. We can see this visually by plotting the eigenspaces below (in dark blue) of the matrix $\mathbf{A}^* = \begin{bmatrix} 1+t & 0 \\ -1 & 1 \end{bmatrix}$ and observing what happens as $t \to 0$:

 $\Rightarrow$  $\Rightarrow$ 

While the first two frames are diagonalizable, the third frame is not, because the two eigenspaces have "folded" on top of one another.

2 Frequently Asked Conceptual Questions

1. **It is possible to have eigenvalues of 0? How should this be interpreted?**

   In fact, it is possible to get eigenvalues of 0. If we set up the equation $\mathbf{A}\mathbf{x} = 0\mathbf{x} = 0$, we observe that if the square matrix $\mathbf{A}$ has a non-trivial (contains more vectors than $\mathbf{0}$) null space, then the dimension of this null space is the multiplicity of the $\mathbf{0}$ eigenvalue.

As an example consider the matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which has a null space of dimension 2. Calculating its characteristic polynomial gives us the equation $\det(\mathbf{A} - \lambda \mathbf{I}_3) = \lambda^2(1-\lambda) = 0$. Its eigenvalues are $\lambda = 1$ with multiplicity 1 and $\lambda = 0$ with multiplicity 2.

3 Examples

N.B: The examples below are more conceptually oriented, because they tend to be the ones that students have difficulty with. For some calculation-oriented examples, see the textbook.

1. **Consider the matrix $\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}$.**

   (a) Determine the eigenvalues of $\mathbf{A}$ and their multiplicities.

   The characteristic polynomial of $\mathbf{A}$ is $(\lambda - 3)^2$, so $\mathbf{A}$ has only the eigenvalue $\lambda = 3$ with multiplicity 2.

   (b) Is the matrix $\mathbf{A}$ defective (not diagonalizable)? If so, explain why; otherwise, write down the $\mathbf{P}$ and $\mathbf{D}$ matrices such that $\mathbf{A} = \mathbf{PDP}^{-1}$.

   To verify that bvA is diagonalizable, we must calculate bases for the null space of $\mathbf{A} - \lambda \mathbf{I_2}$ for$\lambda = 3$. This means we must find a basis for all solutions to $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$, which is simply the set of vectors given by $c \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. However, this means that the dimension of the eigenspace for $\lambda = 3$ is less than the multiplicity of $\lambda = 3$, which is 2. This means that $\mathbf{A}$ is defective and cannot be diagonalized.

2. **Suppose we want to calculate a large power of a matrix $\mathbf{A}^n$ (this is the same as multiplying the matrix A by itself n times). Do the following:**

   (a) Show that if $\mathbf{A}$ is diagonalizable, then we can calculate any power of $\mathbf{A}$ with minimal work because if $\mathbf{A} = \mathbf{PDP}^{-1}$, then $\mathbf{A}^n = \mathbf{PD}^n\mathbf{P}^{-1}$.

   While this looks complicated, the result simply follows from the fact that the $\mathbf{P}$ matrices cancel:

   $\mathbf{A}^n = \mathbf{AAA}...\text{ (n times) } = (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})...\text{ (n times) } = \mathbf{PD}^n\mathbf{P}^{-1}$

Observe that since $\mathbf{D}$ is diagonal, then $\mathbf{D}^n$ is simply the diagonal entries of $\mathbf{D}$ raised to the nth power. The product $\mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$ is much easer to calculate than the matrix product $\mathbf{A}^n$.

(b) Use this trick to come up with a formula for the entries of $\mathbf{A}^n$ if $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$.

We observe that $\mathbf{A}$ is diagonalizable where:

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1 \\ 1/2 & 0 \end{bmatrix}$$

From the trick above, we get:

$$\mathbf{A}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}^n \begin{bmatrix} 1/2 & 1 \\ 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1 \\ 1/2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 \\ (3^n - 1) & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 1 \\ 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ (3^n - 1)/2 & 3^n \end{bmatrix}$$

Additional References:

I would highly recommend looking into the following resources:

1. *Linear Algebra and Its Applications, 5th Edition* by Lay and McDonald (ISBN-13: 978-0321982384)

2. 3Blue1Brown *Essence of Linear Algebra Series*: www.3blue1brown.com/essence-of-linear-algebra-page

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