MTH 2311 Linear Algebra
Week 10 Resources

Colin Burdine

4/13/20

Major Topics:

1. Eigenvalues and Eigenvectors (Continued)
2. Characteristic Equations

Textbook Material:

Linear Algebra and Its Applications, 5th Edition by Lay and McDonald
Sections 4.2-4.4

---

1 Conceptual Review

1.1 Eigenvalues and Eigenvectors (Continued)

Last week we introduced the idea of eigenvalues and eigenvectors. Each eigenvalue λ of a matrix $\mathbf{A}$ is associated with a set of eigenvectors $\mathbf{v}$, all of which satisfy the equation:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}.$$

As we recall, transforming an eigenvector $\mathbf{v}$ by the matrix $\mathbf{A}$ is equivalent simply to multiplying the vector $\mathbf{v}$ by a scalar. To summarize, for any eigenvector $\mathbf{v}$ of $\mathbf{A}$, transforming $\mathbf{v}$ by $\mathbf{A}$ does not change the direction of $\mathbf{v}$, only the magnitude.

Last week we also covered the methods for solving for eigenvectors and eigenvalues, which are summarized below (for a more in-depth description of these calculations, see last week's reference or the corresponding sections in the textbook):

1.2 Finding the Eigenvalues of a matrix (Review)

1. Set $(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$.
2. Solve for λ by setting $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$ to obtain a characteristic polynomial.
3. Find all roots λ of the characteristic polynomial.

1.3 Finding the Eigenvectors of an Eigenvalue (Review)

1. For an eigenvalue λ, set $(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = 0$.
2. Solve for the basis vectors of the null space of the matrix $(\mathbf{A} - \lambda \mathbf{I})$.
3. These basis vectors (and all vectors in their span) are the eigenvectors for λ.

Now that we know how to solve for eigenvalues and eigenvectors, let's try to understand a bit more deeply what these are. A simple question that one might want to pose is: *How many eigenvalues does a given matrix have?* First of all, we know that only square matrices can have eigenvalues. So, for some matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$, we know that all eigenvalues of $\mathbf{A}$ are solutions to $\det(\mathbf{A} - \lambda \mathbf{I_n}) = 0$, which simplifies to a polynomial of order exactly n. From our our pre-calculus days, we might recall the following useful result:

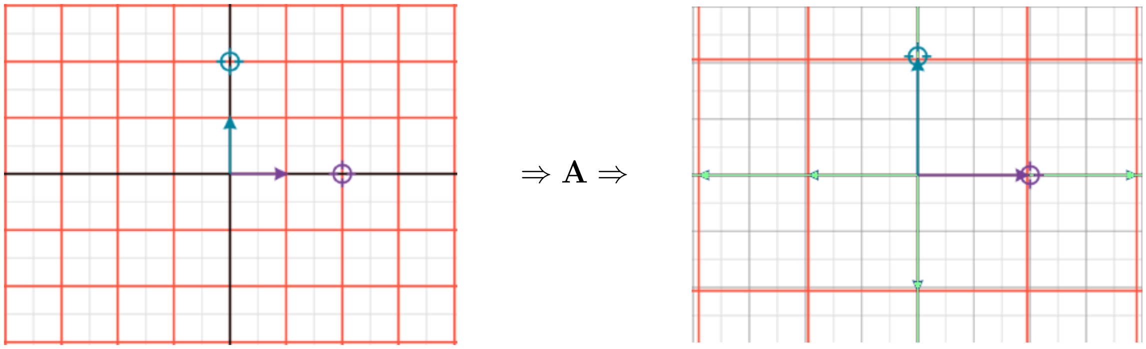
The Fundamental Theorem of Algebra:
A polynomial of order n has at most n real roots. If we allow for roots to be complex-valued (have an imaginary component) or to be counted multiple times (such as -1 is a repeated root of $x^2 + 2x + 1$), then the polynomial has *exactly* n roots.

Thus, if we allow for complex eigenvalues and repeated eigenvalues, then a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ must have *exactly* n eigenvalues. In this course, we will not be working with complex eigenvalues (See the FAQ section below); however, we will encounter repeated eigenvalues. Another case that is often easy to miss is the fact that we can also have eigenvalues of 0, which are also addressed in the FAQ section below.

The idea of repeated eigenvalues may seem a bit mysterious at first, yet simple matrices often have repeated eigenvalues. We can detect the existence of repeated eigenvalues by examining the power of of each root when the characteristic equation of a matrix is factored. This power is referred to as the *multiplicity* of an eigenvalue (Some textbooks use the more specific term *algebraic multiplicity*). Consider, for example the following matrix and its characteristic equation:

$$\mathbf{A} = 2\mathbf{I}_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad \Rightarrow \qquad (\lambda - 2)^2 = 0$$

The characteristic equation in its factored form has a single term $(\lambda - 2)$ with power 2. This indicates that 2 is a repeated eigenvalue, having multiplicity 2. The multiplicity of an eigenvalue isn't just some descriptive number; it actually tells us *the maximum number of linearly independent eigenvectors that can associated with a particular eigenvalue.* We can see this visualized in below with the matrix $\mathbf{A} = 2\mathbf{I}_2$, where the eigenvector bases are highlighted in green:



Notice here that there are two linearly independent eigenvalue bases (in this case the natural bases $\mathbf{e}_1$ and $\mathbf{e}_2$) because the multiplicity of the eigenvector $\lambda = 2$ is 2. However, there are some matrices where there might be fewer linearly independent eigenvectors than the multiplicity of the corresponding eigenvalue.

2 Frequently Asked Conceptual Questions

1. **I'm solving for the eigenvalues of a matrix, but it looks like the characteristic equation does not have any roots. Is this possible?**

   If you ever get non-real roots to your characteristic equation, it is always a good idea to go back to your determinant calculations and make sure that you derived the proper character equation. Most of the matrices that you will be working with in these sections of the book have only real eigenvalues; however, *it is possible for a real-valued matrix to have imaginary eigenvalues.* In this case, you will need to find the imaginary roots by hand (if the characteristic equation is quadratic) or use a root-finding software package.

   If you want to create matrices that are always guaranteed to have real eigenvalues for extra practice, you could use symmetric matrices (matrices where $\mathbf{A} = \mathbf{A}^T$).

2. **It is possible to have eigenvalues of 0? How should this be interpreted?**

   In fact, it is possible to get eigenvalues of 0. If we set up the equation $\mathbf{A}\mathbf{x} = 0\mathbf{x} = 0$, we observe that if the square matrix $\mathbf{A}$ has a non-trivial (contains more vectors than $\mathbf{0}$) null space, then the dimension of this null space is the multiplicity of the $\mathbf{0}$ eigenvalue. As an example consider the matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   which has a null space of dimension 2. Calculating its characteristic polynomial gives us the equation $\det(\mathbf{A} - \lambda\mathbf{I}_3) = \lambda^2(1 - \lambda) = 0$. Its eigenvalues are $\lambda = 1$ with multiplicity

1 and $\lambda = 0$ with multiplicity 2. Interestingly, the number of eigenvectors associated with the 0 eigenvalue is *always* equal to the multiplicity. (Recall that in general, we are only guaranteed to have the dimension of the eigenspace less than or equal to the multiplicity).

3 Examples

N.B: The examples below are more conceptually oriented, because they tend to be the ones that students have difficulty with. For some calculation-oriented examples, see the textbook.

1. **Suppose a square matrix $\mathbb{A}$ has the factored characteristic polynomial:**

$$\lambda^3(\lambda - 4)^2(\lambda + 1)(\lambda - 3)^4 = 0$$

Answer the following:

(a) What is the size of $\mathbf{A}$?

Since the sums of the multiplicities is $3 + 2 + 1 + 4 = 10$, then $\mathbf{A}$ must be a 10×10 matrix.

(b) What are the eigenvalues and their multiplicities of $\mathbf{A}$?

The eigenvalues are $\lambda = 0$ (multiplicity 3), $\lambda = 4$ (multiplicity 2), $\lambda = -1$ (multiplicity 1), and $\lambda = 3$ (multiplicity 4).

(c) What is the nullity of $\mathbf{A}$ (dimension of the null space)? What is the rank of $\mathbf{A}$?

Since the multiplicity of the 0 eigenvalue is 3, then the dimension of the null space of $\mathbf{A}$ must be three (see the FAQ section above). Since the rank of $\mathbf{A}$ plus the nullity of $\mathbf{A}$ must equal 10 (the number of columns of $\mathbf{A}$), then the rank must be 7.

(d) Suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of $\mathbf{A}$ such that $\mathbf{A}\mathbf{v}_1 = 4\mathbf{v}_1$ and $\mathbf{A}\mathbf{v}_2 = 4\mathbf{v}_2$. Is $(\mathbf{v}_1 + \mathbf{v}_2)$ an eigenvector of $\mathbf{A}$? If both $\mathbf{v_1}$ and $\mathbf{v_2}$ are nonzero vectors, is it possible that $\mathbf{v}_1^T\mathbf{v}_2 = 0$ (they are orthogonal)?

Answering the first part, we see that $\mathbf{A}(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{A}\mathbf{v}_1 + \mathbf{A}\mathbf{v}_2 = 4\mathbf{v}_1 + 4\mathbf{v}_2 = 4(\mathbf{v}_1 + \mathbf{v}_2)$, so $(\mathbf{v}_1 + \mathbf{v}_2)$ is by definition an eigenvector of $\mathbf{A}$, associated with the eigenvalue $\lambda = 4$.

For the second part, we observe that because the multiplicity of the eigenvalue $\lambda = 4$ is 2, then the dimension of the set of associated eigenvectors has at most dimension 2, since it is spanned by at most 2 linearly independent eigenvectors. (Notice that this set forms a 2-dimensional subspace of $\mathbb{R}^{10}$). Therefore it is possible (but by no means guaranteed) that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal.

Additional References:

I would highly recommend looking into the following resources:

1. *Linear Algebra and Its Applications, 5th Edition* by Lay and McDonald (ISBN-13: 978-0321982384)
2. 3Blue1Brown *Essence of Linear Algebra Series*: www.3blue1brown.com/essence-of-linear-algebra-page