# MTH 2311 Linear Algebra
# Week 3 Resources

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**Major Topics:**

1. Matrices as Linear Transformations (Continued)
2. Matrix Operations and Properties

**Textbook Material:**

*Linear Algebra and Its Applications, 5th Edition* by Lay and McDonald
**Sections 1.8-2.3**

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# 1 Conceptual Review

## 1.1 Matrices as Linear Transformations (Continued)

Recall last week that we discussed the idea of linearity. Be sure that you have memorized the two properties of linear transformations, because they provide much insight into the properties of matrix operations. These properties are:

1. $f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$
2. $f(\alpha\mathbf{x}) = \alpha f(\mathbf{x})$

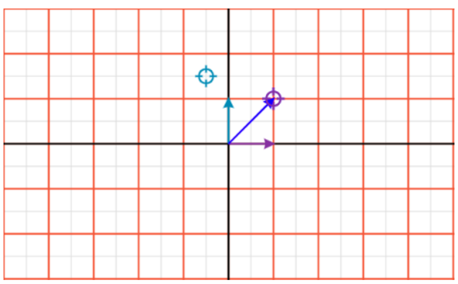
We learned last week that every linear function on a finite vector space can be described by some matrix $\mathbf{A}$. Thus, we can re-write the properties above in the lanquage of matrices:

1. $\mathbf{A}(\mathbf{x}+\mathbf{y}) = \mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{y}$
2. $\mathbf{A}(\alpha\mathbf{x}) = \alpha\mathbf{A}\mathbf{x}$

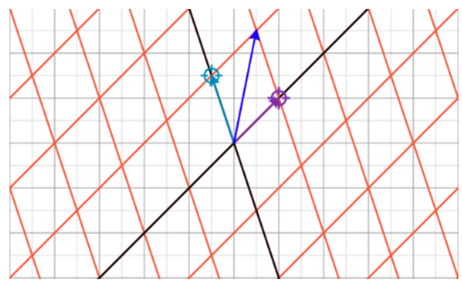
An excellent way to explain these properties to students is to use a geometric example. Suppose we have the following matrix $\mathbf{A}$ and the vectors $\mathbf{x}$ and $\mathbf{y}$:

$$\mathbf{A} = \begin{bmatrix} 1 & -0.5 \\ 1 & 1.5 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; \mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

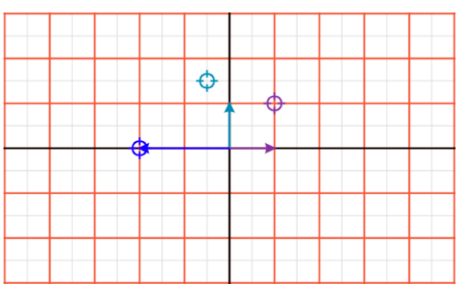
We can observe that the first property, $\mathbf{A}(\mathbf{x}+\mathbf{y}) = \mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{y}$, holds true:



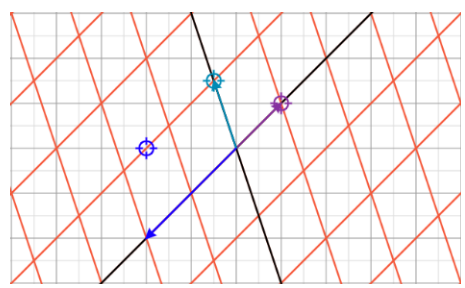
$\rightarrow \mathbf{A}(\mathbf{x}+\mathbf{y}) \rightarrow$



Similarly, for $\alpha = -2$, we can observe that the second property, $\mathbf{A}(-2\mathbf{x}) = -2\mathbf{A}\mathbf{x}$, also holds:



$\rightarrow \mathbf{A}(-2\mathbf{x}) \rightarrow$



Using geometric examples is an excellent way to build up your intuition and understanding of linear transformations, expecially since you will be moving on to more complex matrix operations. Having a solid visual understanding of linear transformations will definitely help you in the coming chapters.

## 1.2 Matrix Operations

While most of the matrix operations (such as inversion, determinant, adjugate, etc.) are more mechanical than conceptual, I will not provide a detailed breakdown of how to perform the operations here. If you need a review of how to do these calculations, see the *Linear*

*Algebra Reference Sheet*. However, I will provide a brief overview of some of the common misconceptions in the FAQ section below.

# 2 Frequently Asked Conceptual Questions

1. **Why is regular multiplication commutative, but not matrix multiplication?**

   Matrix multiplication is not commutative, contrary to what one might expect when trying to work with matrices symbolically. With non-square matrices, this is obvious, but with square matrices it is less so. Whenever you (or your students) are working with matrix equations, be sure that the order of multiplication is preserved from left to right. When multiplying both sides of an equation by a matrix, the expressions on either side must maintain their order of multiplication. There are some exceptions to this rule (such as with square diagonal matrices), however. While matrix multiplication is non-commutative, it is associative, meaning that the the groupings of parentheses in a series of multiplications can be reordered freely, so long as the left-to-right order is preserved:
   $$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$$

2. **How do I figure out if a matrix has an inverse?** One method of doing this is to use the determinant, which will be introduced next week. Alternatively, one could consider a matrix $\mathbf{A}$ to be invertible if and only if the equation $\mathbf{Ax} - \mathbf{0}$ only has the trivial solution ($\mathbf{x} = \mathbf{0}$). If this homogeneous equation has only the trivial solution, then the linear transformation encoded by $A$ is not one-to-one, since there are multiple solutions to the homogeneous system. Furthermore, one could also detect that $A$ is not invertible by trying to invert it through row-reduction and observing that the identity matrix is not obtained on the lefthand side. For example, if we try to row reduce the non-invertible matrix below, we do not end up with the identity matrix on the lefthand partition of the augmented matrix:

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \quad \sim \quad \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right] \rightarrow \text{Not Invertible}$$

# 3 Examples

**N.B: The examples below are more conceptually oriented, because they tend to be the ones that students have difficuly with. For some calculation-oriented examples, see the textbook.**

1. **Suppose** $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$**. Show that** $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

   Using the augmented matrix method, we row-reduce $[\mathbf{A}|I_2]$ to get:

$$\left[\begin{array}{cc|cc} a & b & 10 & \\ c & d & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & (d - bc/a) & (-c/a) & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} a & 0 & \frac{ad}{ad-bc} & \frac{-ab}{ad-bc} \\ 0 & (d - bc/a) & (-c/a) & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \qquad \rightarrow \qquad \mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

2. **Suppose for some matrix A, the equation $\mathbf{Ax} = \mathbf{b}$ has infinitely many solutions. Is A invertible?**

   No, $\mathbf{A}$ is not invertible. We can justify this with the following:

   Suppose that $\mathbf{A}$ is invertible. Then there exists a matrix $\mathbf{A}^{-1}$ so that $\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ This means that $\mathbf{x}$ can only be $\mathbf{A}^{-1}\mathbf{b}$, which must evaluate to be a single vector. This contradicts the fact that $\mathbf{Ax} = \mathbf{b}$ has infinitely solutions. Hence, $\mathbf{A}$ cannot possibly be invertible.

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## Additional References:

I would highly recommend looking into the following resources:


1. *Linear Algebra and Its Applications, 5th Edition* by Lay and McDonald (ISBN-13: 978-0321982384)
2. 3Blue1Brown *Essence of Linear Algebra Series*: www.3blue1brown.com/essence-of-linear-algebra-page


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