# MTH 2311 Linear Algebra
# Week 2 Resources

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**Major Topics:**

1. Vector and Matrix Equations (Continued)
2. Span and Linear Independence
3. Matrices as Linear Transformations

**Textbook Material:**

*Linear Algebra and Its Applications, 5th Edition* by Lay and McDonald
**Sections 1.4-1.9**

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# 1 Conceptual Review

## 1.1 Vector and Matrix Equations (Continued)

We left off last week with a general review of the basic matrix operations and forms, such as the matrix-vector product, dot (inner) product, and the matrix-matrix product. When we use these forms and operations to solve some problem, we typically do it in the form of the matrix-vector equation $\mathbf{Ax} = \mathbf{b}$. However, there are several interpretations of this matrix-vector equation that students should be aware of and able to use interchangably. In this course, there are 4 interpretations of the equation $\mathbf{Ax} = \mathbf{b}$ that we will use interchangably. These interpretations are:

1. A matrix-vector product as written in the form $\mathbf{Ax} = \mathbf{b}$

2. A system of linear equtions involving the variables that comprise the entries of $\mathbf{x}$

3. A single vector equation consisting of linear combinations of the columns of $\mathbf{A}$ equal to the vector $\mathbf{b}$

4. A linear transformation (function) mapping vectors in $\mathbb{R}^n$ to $\mathbb{R}^m$ encoded by the $m \times n$ matrix $\mathbf{A}$

To help make these interpretations more concrete, let's consider the simple $2 \times 2$ matrix equation below:

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \mathbf{b}$$

The equation is written in the form of (1) in the list above, yet we can also interpret this as a system of linear equations, which we covered last week. This is interpretation number (2) in the list above:

$$\begin{cases} x_1 & + & 2x_2 & = & b_1 \\ 2x_1 & + & -x_2 & = & b_2 \end{cases}$$

However, if we write out the product of the $\mathbf{A}\mathbf{x} = \mathbf{b}$ equation above, in vector format, we get the following:

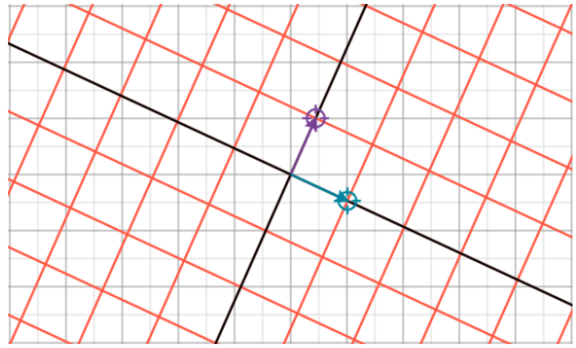
$$\mathbf{A}\mathbf{x} = \mathbf{b} \to \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \to \begin{bmatrix} (1)x_1 + (2)x_2 \\ (2)x_1 + (-1)x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \to \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ -1 \end{bmatrix} x_2 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \mathbf{b}$$

On the righthand side, we are left with a vector equation, which is interpretation (3) in our list above. This interpretation is particularly useful if we want to conclude anything about the *span* of the columns of the matrix $\mathbf{A}$. We will discuss the concept of span in detail in the section below, but it is important to consider the geometric interpretation of the equation above. This interpretation is most succintly expressed in the following question:

*Can we lengthen the vectors* $[1\ 2]^T$ *and* $[2\ \ -1]^T$ *and add them together tail-to-tip to produce the vector* $[b_1\ b_2]^T$*?*

One way of answering this question is to consider an alternative coordinate system in which we treat the $x$ axis like the vector $[2\ \ -1]^T$ and the $y$ axis like the vector $[1\ 2]^T$. Drawing gridlines parallel and perpendicular to these vectors spaced $\sqrt{5} = \|\ [2\ \ -1]^T\| = \|[1\ 2]^T\|$ units apart, we obtain the grid shown in red below:

$$\begin{bmatrix}1\\2\end{bmatrix}x_1 + \begin{bmatrix}-1\\2\end{bmatrix}x_2 = \begin{bmatrix}b_1\\b_2\end{bmatrix} = \mathbf{b}$$



*A geometric interpretation of the vector equation*

Using the illustration as a guide, it becomes clear that the answer to the question that we posed above is *yes*. For any 2D vector $[b_1\ b_2]^T$, we simply its corresponding position using our new "x" and "y" axis grid shown in red. This simple illustration is also quite useful in explaining interpretation (4) to students. However, we will defer the details of this interpretation to section 1.3 below.

## 1.2 Span and Linear Independence

The concept of span is something that students often struggle with when they begin working with geometric interpretations of matrices. However, in order to understand span, it is important that students have a solid grasp on the notion of *linear independence*.

A set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, ...\}$ are said to be linearly independent if no vector $\mathbf{v_i}$ in the set can be written as some linear combination of the other vectors in the set. Using vector notation, this is equivalent to showing that there do not exist values $x_1$, $x_2$, $x_3$, ... so that $x_1\mathbf{v_1} + x_2\mathbf{v_2} + x_3\mathbf{v_3} + ... = \mathbf{v_i}$. At first this definition seems a bit confusing, and it seems to suggest that showing the linear independence of a set of vectors a difficult task to accomplish. However, by subtracting $v_i$ from the right side of the equation, we get an equivalent definition that is less intuitive, but easier to work with:

*A set of vectors* $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, ...\}$ *is linearly independent if the only solution to the equation* $x_1\mathbf{v_1} + x_2\mathbf{v_2} + x_3\mathbf{v_3} + ... = 0$ *is* $x_1 = x_2 = ... = 0$*. If there exists another solution apart from* $x_1 = x_2 = ... = 0$*, we instead say that the set of vectors is linearly dependent.*

Upon closer examination, we see that this in an instance of interpretation (3) in the list of interpretations we generated above. In fact, we can convert this vector equation directly to a matrix-vector product of the form $\mathbf{Ax} = \mathbf{b}$, which is interpretation (1) above. All we need to do is put the vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, ..$ into the columns of $A$, so that we are left the following equation:

$$\mathbf{Ax} = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & ... \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ ... \end{bmatrix} = \mathbf{0}$$

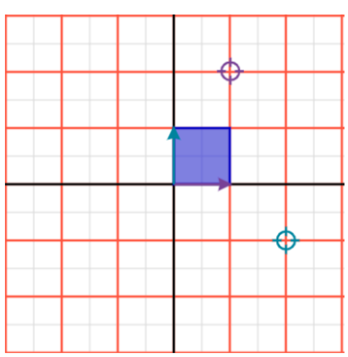
Thus, to test the linear independence of $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, ...\}$, we simply row-reduce the above matrix and determine if there exists a solution that is not $\mathbf{x} = \mathbf{0}$. Such a nonzero solution is usually called a *non-trivial solution* in Linear Algebra.

## 1.3 Matrices as Linear Transformations

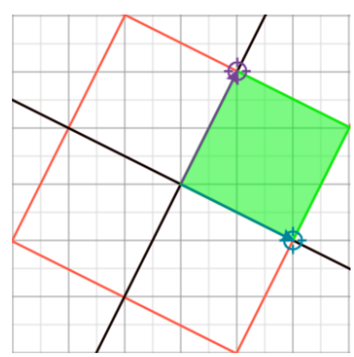
Matrices are very useful structures for representing linear transformations. But by now, most students tend to ask the question "what exactly *is* a linear transformation?". The textbook definition of a linear transformation is a function $f$ that has the following two properties:

1. $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$
2. $f(\alpha\mathbf{x}) = \alpha f(\mathbf{x})$

Note that most students seeing these definitions for the first time will think that $f$ is merely a function that maps one real number to another real number, or as mathematicians write: $f : \mathbb{R} \to \mathbb{R}$. But in the definition above $\mathbf{x}$ is a vector, not just a number! (observe that a number can be thought of as 1D vector). In truth, a linear transformation can actually involve vectors of real numbers, which mathematicians denote by $f : \mathbb{R}^m \to \mathbb{R}^n$ where $m$ and $n$ are positive integers. In our $2 \times 2$ example, matrix above, we observed that the transformation resembled a simultaneous flip, rotation and scaling that mapped $\mathbb{R}^2 \to \mathbb{R}^2$:



$$\to \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to$$



*A linear transformation from* $\mathbb{R}^2 \to \mathbb{R}^2$

This begs the question- *what kinds of linear functions* $f : \mathbb{R}^n \to \mathbb{R}^m$ *exist?*. The answer, suprisingly, is that every single transformation mapping $\mathbb{R}^n \to \mathbb{R}^m$ can be represented uniquely by a $m \times n$ matrix! This concept is perhaps the most crucial takeaway of Linear Algebra, so be sure that your students are aware of it!

So we know that linear transformations can be encoded by matrices, it simply remains to show how we *interpret* these transformations and discover their properties. This brings us to our final (and perhaps most important) interpretation of matrices as linear transformations (4). In the units to come, students will be learning more about these properties.

## 2 Frequently Asked Conceptual Questions

1. **I heard that the span of a set of vectors in $\mathbb{R}^n$ is $\mathbb{R}^n$. What does this mean?**

   If the span of a set of vectors is, for the sake of illustration, in $\mathbb{R}^2$, then that would mean that *every* 2D vector can be written as a linear combination of the vectors in the set. Intuitively, to get a set of vectors in $\mathbb{R}^n$ whose span is $\mathbb{R}^n$ we need at least $n$ vectors. One vector set that is guaranteed to span $\mathbb{R}^n$ (for any $n$) is the set of *natural basis vectors*. These vectors are simply the $n$ columns of the $n \times n$ identity matrix, which one can think of as each representing an axis ($x$, $y$, $z$, etc. ), each of which is orthogonal to all the others.

2. **Are there any tips for getting better at elimination? I keep messing up the row operations.**

   Row reduction is a complex process- the only way to get better at it is to practice consistently. Try reducing matrices of different sizes (within reason, of course) and of different solution type (no solution, one solution, infinite solutions). Varied practice done at consistent intervals is a good recipe for success.

## 3 Examples

**N.B: The examples below are more conceptually oriented, because they tend to be the ones that students have difficuly with. For some calculation-oriented examples, see the textbook.**

1. **Show that if $\mathbf{v}$ is a solution to $\mathbf{Av} = \mathbf{0}$ and $\mathbf{x}$ is a solution to $\mathbf{Ax} = \mathbf{b}$, then $\mathbf{A(x + v) = 0}$.**

   Since multiplying a vector by $A$ is a linear transformation, then:

$$\mathbf{A(x + v) = Ax + Av = (b) + (0) = b}$$

2. **Suppose we have the matrix A and the vector b as shown below:**

$$A = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -2 & -16 \\ 2 & 1 & 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

**Let $\mathbf{x} = \mathbf{v}$ be one solution to $\mathbf{Ax} = \mathbf{b}$. Find all solutions to $\mathbf{Ax} = \mathbf{b}$ in terms of the known solution $\mathbf{v}$**

Using the property found above in Exercise (1), we only need to find the solutions to the homogeneous matrix equation $\mathbf{Ax} = \mathbf{0}$. Row reducing $\mathbf{A}$, we get the matrix $\mathbf{A}'$:

$$\mathbf{A}' = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \end{bmatrix},$$

so the solutions to $\mathbf{A}'\mathbf{x} = \mathbf{0}$ are of the form:

$$\mathbf{x} = \begin{bmatrix} 2 \\ -7 \\ 1 \end{bmatrix} t, \qquad \text{where } t = x_3 \text{ is a free variable in } \mathbb{R}.$$

We can apply the property discovered in Exercise 1 to get all solutions to $\mathbf{Ax} = \mathbf{b}$, which are of the form:

$$\mathbf{x} = \mathbf{v} + \begin{bmatrix} 2 \\ -7 \\ 1 \end{bmatrix} t$$

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## Additional References:

I would highly recommend looking into the following resources:


1. *Linear Algebra and Its Applications, 5th Edition* by Lay and McDonald (ISBN-13: 978-0321982384)

2. 3Blue1Brown *Essence of Linear Algebra Series*: www.3blue1brown.com/essence-of-linear-algebra-page


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