MTH 1320 – PRECALCULUS
FALL 2020 WEEK 11 RESOURCE

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Welcome back, Precalculus scholars! This resource focuses on material covered in the eleventh week of classes, namely angles, the unit circle, sine, and cosine from sections 5.1-5.2 of OpenStax's *Precalculus*. Please refer to previous resources if you would like to review other topics. Don't forget to sign up for Group Tutoring on Tuesdays at 5:00!

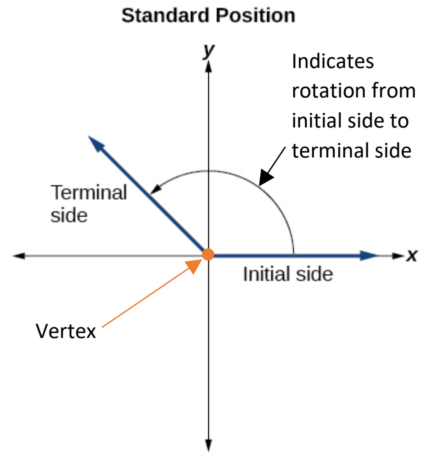

Figure 2. An Angle in Standard Position
Source: Adapted from [1]

5.1 Angles

Angles are a foundational concept for trigonometric functions, the topic of Chapter 5.

Definitions and Conventions

An **angle** is "the union of two rays having a common endpoint" [1]. Each **ray** starts at the endpoint and extends in a straight line from it out to infinity [1]. The endpoint is also known as the angle's **vertex** [1]. When we draw angles, it is the convention to draw them in **standard position**, in which the vertex lies at the origin of the coordinate plane and the initial side lies on the positive *x*-axis [1]. See Figure 1. Also, angles can be positive or negative. **Positive angles** are measured in the counterclockwise direction, and **negative angles** are measured in the clockwise direction [1]. For example, the angle in Figure 1 is a positive angle because the arc with the arrow is drawn in the counterclockwise direction.

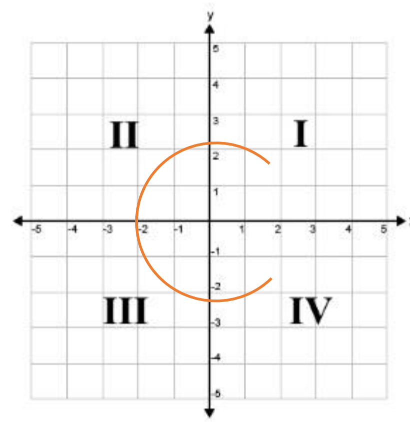

Figure 1. Quadrants of the Coordinate Plane
Source: Adapted from [2]

Another important convention to know when discussing trigonometry is the four quadrants of the coordinate plane. The quadrants are numbered in the counterclockwise direction as shown in Figure 2. (I like to remember them as a "c" shape.)

Drawing and Measuring Angles

The **measure of an angle** is "the amount of rotation from the initial side to the terminal side" (marked with the arc with the arrow like in Figure 1) [1]. There are two units used to measure angles: degrees and radians. A **degree** is "1/360 of a circular rotation" and an angle measure in degrees is marked with the symbol ° [1]. A **radian** is "the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle" [1]. See Figure 3. This means that when a 1-radian angle is drawn in standard position, and the tips of the rays touch a circle centered around the origin, the arc connecting the initial and terminal sides of the angle is equal in length to the radius of the circle.

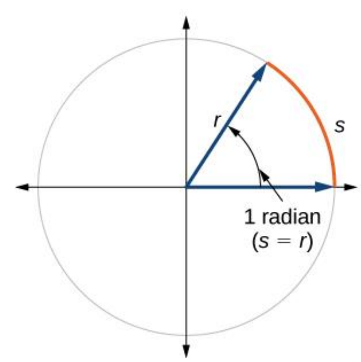

Figure 3. Measuring Radians
Source: Adapted from [1]

We could prove that one full rotation is $360°$ or 2π radians. Therefore,

$$180° = \pi \text{ radians.}$$

We can use the above equality to convert between degrees and radians. Note that if an angle measure is not given a unit, it is implied that the units are radians [1].

To draw an angle in standard position, use the following steps:

1. Divide the given angle measure by $360°$ or 2π radians, depending on the given units.
2. Reduce the fraction.
3. Rewrite the reduced fraction so that you can visualize the portion of the circle that it represents.
4. Draw the initial side of the angle on the positive *x*-axis.
5. Rotate counterclockwise if the angle is positive and clockwise if the angle is negative.
6. Draw the terminal side so that the angle contains the fraction that you calculated.

Coterminal Angles

Because angles rotate in a circle, once we rotate counterclockwise past $360°$ or 2π radians or clockwise past $0°$ or 0 radians, the angle is equivalent to an angle between $0°$ (0 radians) and $360°$ (2π radians). These equivalent angles are called coterminal. **Coterminal angles** are "two angles in standard position that have the same terminal side" [1]. For a particular angle that is not between $0°$ (0 radians) and $360°$ (2π radians), we often want to find the coterminal angle between $0°$ (0 radians) and $360°$ (2π radians) because this range is easy to work with. If we are given an angle greater than $360°$ (2π radians), we can find the coterminal angle between $0°$ (0 radians) and $360°$ (2π radians) by subtracting $360°$ (2π radians) from the given angle until the angle is less than $360°$ (2π radians). If we are given an angle less than $360°$ (2π radians), we can find the coterminal angle between $0°$ (0 radians) and $360°$ (2π radians) by adding $360°$ (2π radians) to the given angle until the angle is greater than $0°$ (0 radians).

Reference Angles

An angle's **reference angle** is "the size of the smallest acute angle, t', formed by the terminal side of the angle t and the horizontal axis" [1]. A reference angle drawn in standard position will always be in quadrant I of the coordinate plane. The calculation of an angle's reference angle varies based on the quadrant in which the original angle lies. See Figure 4.

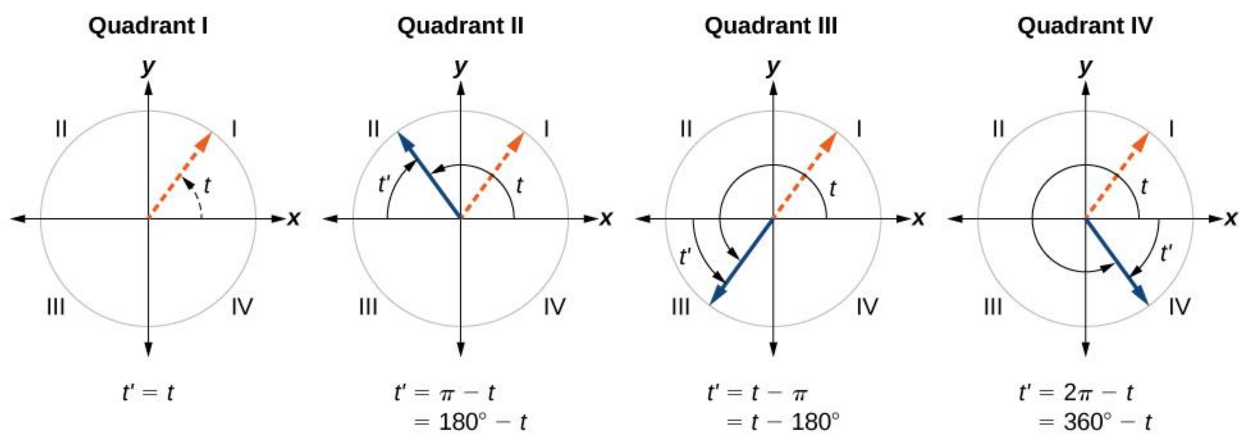

Figure 4. Calculating Reference Angles of Angles in Different Quadrants
Source: [1]

Applications of Angles

Angles can be applied to many applications involving circles and rotational motion.

Arc Length

An **arc** is a portion of the outline of a circle. The formula for arc length s is

$$s = r\theta,$$

where r is the radius of the circle that the arc is part of, and θ is the measure in radians of the angle that forms the arc. See Figure 5.

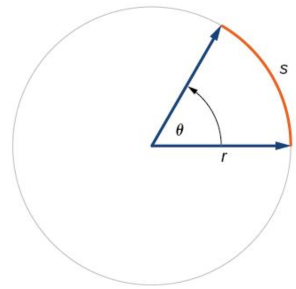

Figure 5. Arc Length
Source: [1]

Area of a Sector

A **sector** is "a region of a circle bounded by two radii and the intercepted arc, like a slice of pizza or pie" [1]. See Figure 6. To find a sector's area, we can multiply the whole circle's area (πr^2) by the fraction of the circle that the sector is. This results in the formula:

$$A = \frac{1}{2}\theta r^2.$$

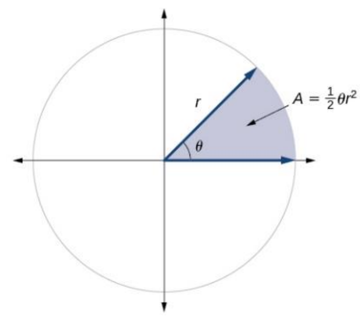

Figure 6. Area of a Sector
Source: [1]

Note that θ must be in radians for the equation to be valid.

Linear and Angular Speed

When an object moves in a circle, it is said to be in **rotational motion**. An object in rotational motion has **linear speed** v or "speed along a straight path" like all objects in motion. However, it also has **angular speed** ω, which only objects in rotation motion have. Table 1 gives the formulas for linear speed, angular speed, and the relationship between the two.

Table 1. Linear and Angular Speed

Linear Speed	$v = \frac{s}{t}$
Angular Speed	$\omega = \frac{\theta}{t}$
Relationship Between Linear and Angular Speed	$v = r\omega$

Note that s is in general the distance travelled by the object, but for circular motion it is also the arc length. t is the time elapsed, θ is the angle traversed, and r is the radius of the circular path. Also, note that ω is the lowercase Greek letter omega and should be in radians per unit time for the third equation in Table 1 to hold.

5.2 Unit Circle: Sine and Cosine Functions

The **unit circle** is the circle with radius 1 centered at the origin. Figure 7 shows the unit circle with frequently used angles and the *x*- and *y*-coordinates were the angles' terminal sides intersect the circle. Additionally, it is annotated with the signs of the *x* and *y* values in each quadrant. It is probably the most important figure in all of trigonometry. I highly recommend memorizing it, at least the first quadrant, from which all the other quadrants can be found.

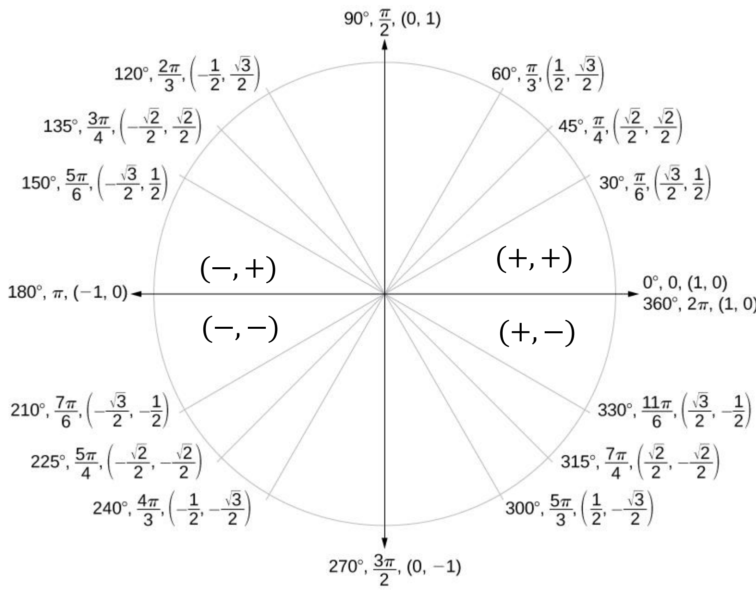

Figure 7. The Unit Circle
Source: Adapted from [1]

If we call one of these angles θ, the point (x, y) at which the terminal ray of the angle intersects the unit circle is given by

$$x = \cos\theta \text{ and } y = \sin\theta.$$

$f(\theta) = \cos\theta$ is the **cosine function**, and $f(\theta) = \sin\theta$ is the **sine function**. Note that these functions' domain is all real numbers, and their range is $-1 \leq \theta \leq 1$.

An important identity relating sine and cosine is the **Pythagorean Identity**:

$$\cos^2\theta + \sin^2\theta = 1.$$

This identity comes from the equation for the unit circle, $x^2 + y^2 = 1$ [1].

Reminders

Thank you for reading this resource! Here are a few reminders about other Precalculus opportunities:

- Group Tutoring for Precalculus is on Tuesdays from 5:00 to 6:00 p.m. through Microsoft Teams. If you have not received an email inviting you to sign up, and you want to attend, please email me at Sydney_Schirner1@baylor.edu, and I will make sure you are included!
- The Tutoring Center has great Precalculus videos on YouTube! You can find links for them at https://www.baylor.edu/support_programs/index.php?id=955694.
- If you want individual help, you can also schedule a FREE 30-minute appointment with one of our awesome Precalculus tutors at https://www.baylor.edu/tutoring.

References

[1] J. Abramson *et al.*, *Precalculus*. Houston, Texas, USA: Rice Univ., 2014. [Online]. Available: https://openstax.org/details/books/precalculus

[2] Help Teaching. "Quadrants of the Coordinate Plane." HelpTeaching.com. Accessed: Oct. 30, 2020. [Online]. Available: https://www.helpteaching.com/lessons/213/quadrants-of-the-coordinate-plane