MTH 1320 – PRECALCULUS
FALL 2020 WEEK 10 RESOURCE

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Welcome back, Precalculus scholars! This resource focuses on material covered in the tenth week of classes, namely exponential and logarithmic properties and equations from sections 4.5-4.6 of OpenStax's *Precalculus*. Please refer to previous resources if you would like to review other topics. Don't forget to sign up for Group Tutoring on Tuesdays at 5:00!

4.5 Logarithmic Properties

When working with exponentials and logarithms, it is important to know their properties. Your textbook does not have a separate section covering properties of exponential functions, so Table 1 summarizes both exponential and logarithmic properties. Because exponentials and logarithms are inverses, it is not surprising that their properties also look inverted.

Table 1. Exponential and Logarithmic Properties

Name of Property	Exponential Version	Logarithmic Version
	$b^0 = 1$	$\log_b(1) = 0$
	$b^1 = b$	$\log_b(b) = 1$
Inverse property	$b^{\log_b x} = x \quad (x > 0)$	$\log_b b^x = x$
One-to-one property	$b^x = b^y$ if and only if $x = y$	$\log_b(M) = \log_b(N)$ if and only if $M = N$
Product rule	$b^x b^y = b^{x+y}$	$\log_b(MN) = \log_b(M) + \log_b(N)$
Quotient rule	$\frac{b^x}{b^y} = b^{x-y}$	$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$
Power rule	$(b^x)^y = b^{xy}$	$\log_b(M^n) = n\log_b(M)$

Expanding and Condensing Logarithmic Expressions

Logarithmic properties can be used to expand or condense logarithmic expressions. To expand a logarithmic expression, you should often use the following order:

1. Quotient rule
2. Product rule
3. Power rule

Note that when you have a ratio of products, an alternate approach is to write every factor in the numerator with a plus sign and every factor in the denominator with a minus sign, as follows:

$$\log_b\left(\frac{M_1 M_2 \dots M_m}{N_1 N_2 \dots N_n}\right) = \log_b(M_1) + \log_b(M_2) + \cdots + \log_b(M_m) - \log_b(N_1) - \log_b(N_2) - \cdots - \log_b(N_n)$$

To condense a logarithmic expression, you should use the reverse of the expansion order [1]:

1. Power rule
2. Product rule
3. Quotient rule

Example 1

Expand $\ln\left(\frac{\sqrt{(x-1)(2x+1)^2}}{(x^2-9)}\right)$. [1]

Method 1:
Use the quotient rule for logarithms:

$$\ln\left(\frac{\sqrt{(x-1)(2x+1)^2}}{(x^2-9)}\right) = \ln\left(\sqrt{(x-1)(2x+1)^2}\right) - \ln(x^2-9)$$

Rewrite the square root as an exponent:

$$= \ln\left(\left((x-1)(2x+1)^2\right)^{1/2}\right) - \ln(x^2-9)$$

Use the power rule of exponents in the first logarithm:

$$= \ln\left((x-1)^{\frac{1}{2}}(2x+1)^1\right) - \ln(x^2-9)$$

Factor the argument of the second logarithm:

$$= \ln\left((x-1)^{\frac{1}{2}}(2x+1)^1\right) - \ln((x-3)(x+3))$$

Use the product rule for logarithms:

$$= \ln\left((x-1)^{\frac{1}{2}}\right) + \ln(2x+1) - [\ln(x-3) + \ln(x+3)]$$

Distribute the negative sign:

$$= \ln\left((x-1)^{\frac{1}{2}}\right) + \ln(2x+1) - \ln(x-3) - \ln(x+3)$$

Use the power rule for logarithms on the first logarithm:

$$= \frac{1}{2}\ln(x-1) + \ln(2x+1) - \ln(x-3) - \ln(x+3)$$

Method 2:
Rewrite the square root as an exponent:

$$\ln\left(\frac{\sqrt{(x-1)(2x+1)^2}}{(x^2-9)}\right) = \ln\left(\frac{\left((x-1)(2x+1)^2\right)^{\frac{1}{2}}}{(x^2-9)}\right)$$

Use the power rule of exponents:

$$= \ln\left(\frac{(x-1)^{\frac{1}{2}}(2x+1)^1}{(x^2-9)}\right)$$

Factor the denominator:

$$= \ln\left(\frac{(x-1)^{\frac{1}{2}}(2x+1)}{(x-3)(x+3)}\right)$$

Expand by writing every factor in the numerator with a plus sign and every factor in the denominator with a minus sign:

$$= \ln\left((x-1)^{\frac{1}{2}}\right) + \ln(2x+1) - \ln(x-3) - \ln(x+3)$$

Use the power rule for logarithms on the first logarithm:

$$= \frac{1}{2}\ln(x-1) + \ln(2x+1) - \ln(x-3) - \ln(x+3)$$

The Change of Base Formula

Another useful tool is the **change of base formula** which says

$$\log_b(M) = \frac{\log_n(M)}{\log_n(b)},$$

where n is a new base. The change of base formula can be used to evaluate logarithms in a calculator that only has functions for the common logarithm ($\log(x)$) and the natural logarithm ($\ln(x)$) [1].

4.6 Exponential and Logarithmic Equations

Now that you know the exponential and logarithmic properties, you can use them to solve exponential and logarithmic equations.

Solving Exponential Equations

There are two main ways to solve exponential equations. Before we discuss them, note that a positive base raised to a power cannot produce a negative number, and a logarithm cannot have a negative argument. Therefore, some exponential equations might have no solution or an **extraneous solution**, "a solution that is correct algebraically but does not satisfy the conditions of the original equation" [1].

Using like Bases

Because of the one-to-one property of exponential functions, we know that if two exponentials are equal and have the same base, their exponents are also equal. We can use this property to solve exponential equations when the exponentials have the same base. If the exponentials do not have the same base, we can sometimes rewrite one of the bases as a power of the other base.

Example 2

Solve $5^{2x} = 25^{3x+2}$. [1]

Rewrite 25 as a power of 5:

$$5^{2x} = (5^2)^{3x+2}$$

Use the power rule of exponents:

$$5^{2x} = 5^{2(3x+2)}$$

Distribute the 2:

$$5^{2x} = 5^{6x+4}$$

Use the one-to-one property of exponential functions:

$$2x = 6x + 4$$

Combine like terms and solve for x:

$$-4x = 4$$
$$x = -1$$

Using Logarithms

If the exponentials cannot be written with the same base or there is only one exponential term, we can still solve the equation using logarithms. Equality is maintained if we take the same logarithm of both sides of the equation. Your textbook recommends using the common logarithm if one of the exponentials has 10 as its base and using the natural logarithm otherwise [1]. Once you have applied the logarithm to both sides of the equation, you can use the properties of logarithms to solve.

Example 3

Solve $2^x = 3^{x+1}$. [1]

Take the natural logarithm of both sides of the equation:

$$\ln(2^x) = \ln(3^{x+1})$$

Use the power rule for logarithms:

$$x\ln(2) = (x+1)\ln(3)$$

Distribute the $\ln(3)$:

$$x\ln(2) = x\ln(3) + \ln(3)$$

Combine like terms and solve for x:

$$x\ln(2) - x\ln(3) = \ln(3)$$

$$x(\ln(2) - \ln(3)) = \ln(3)$$

$$x = \frac{\ln(3)}{\ln(2) - \ln(3)}$$

Use the quotient rule for logarithms to simplify:

$$x = \frac{\ln(3)}{\ln\left(\frac{2}{3}\right)}$$

Solving Logarithmic Equations

There are also two main ways to solve logarithmic equations. Note that logarithmic equations may have extraneous solutions as well, which cause the argument of a logarithm to be negative.

Using the Definition of a Logarithm

Recall that a logarithm has the following definition:

$y = \log_b(x)$ is equivalent to $b^y = x$.

We can often use this fact to solve logarithmic equations by rewriting a logarithm in its exponential form. Another way to look at this is that, just like we can take the logarithm of each side of an equation, we can "exponentiate" each side of an equation and maintain equality.

Example 4

Solve $2\ln(x+1) = 10$. [1]

Method 1:

Isolate the natural logarithm:

$$\ln(x+1) = \frac{10}{2}$$

$$\ln(x+1) = 5$$

Use the definition of the natural logarithm:

$$x + 1 = e^5$$

Isolate the x:

$$x = e^5 - 1$$

Method 2:
Isolate the natural logarithm:

$$\ln(x+1) = \frac{10}{2}$$
$$\ln(x+1) = 5$$

Exponentiate both sides of the equation:

$$e^{\ln(x+1)} = e^5$$

Use the inverse property to cancel the e and the $\ln()$:

$$x + 1 = e^5$$

Isolate the x:

$$x = e^5 - 1$$

Using the One-to-One Property of Logarithms

Just like for exponentials, if we have an equation with two logarithms of the same base, we can use the one-to-one property to solve it. We can set the arguments of the two logarithms equal to each other if the two logarithms are equal.

Example 5

Solve $\ln(x^2) = \ln(1)$. [1]
Use the one-to-one property of logarithms:

$$x^2 = 1$$

Take the square root of both sides of the equation:

$$\sqrt{x^2} = \pm\sqrt{1}$$
$$x = \pm 1$$

Check for extraneous solutions:

$$\ln(1^2) = \ln(1) = 0$$
$$\ln((-1)^2) = \ln(1) = 0$$

Both 1 and -1 are valid solutions.

Reminders

Thank you for reading this resource! Here are a few reminders about other Precalculus opportunities:

- Group Tutoring for Precalculus is on Tuesdays from 5:00 to 6:00 p.m. through Microsoft Teams. If you have not received an email inviting you to sign up, and you want to attend, please email me at Sydney_Schirner1@baylor.edu, and I will make sure you are included!
- The Tutoring Center has great Precalculus videos on YouTube! You can find links for them at https://www.baylor.edu/support_programs/index.php?id=955694.
- If you want individual help, you can also schedule a FREE 30-minute appointment with one of our awesome Precalculus tutors at https://www.baylor.edu/tutoring.

References

[1] J. Abramson *et al*., *Precalculus*. Houston, Texas, USA: Rice Univ., 2014. [Online]. Available: https://openstax.org/details/books/precalculus