MTH 1320 – PRECALCULUS
FALL 2020 WEEK 7 RESOURCE

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Welcome back, Precalculus scholars! This resource focuses on material covered in the seventh week of classes, namely rational functions from Section 3.7 of OpenStax's *Precalculus*. Please refer to previous resources if you would like to review other topics. Don't forget to sign up for Group Tutoring on Tuesdays at 5:00!

3.7 Rational Functions

A **rational function** is "a function that can be written as the quotient of two polynomial functions $P(x)$ and $Q(x)$" [1]. The adjective *ratio*nal is used because it is a *ratio* of polynomials. Recall that a polynomial function has the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$. Therefore, if we have a polynomial $P(x)$ with degree p and coefficients a_i and another polynomial $Q(x)$ with degree q and coefficients b_i, their ratio would be

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_p x^p + a_{p-1}x^{p-1} + \cdots + a_2x^2 + a_1x + a_0}{b_q x^q + b_{q-1}x^{q-1} + \cdots + b_2x^2 + b_1x + b_0}.$$

Note that $Q(x)$ cannot be equal to zero because the function would be undefined for all values of x.

Arrow Notation

To discuss the behavior of rational functions, it is important to understand **arrow notation**. We have used arrow notation previously to discuss functions' end behavior (as $x \to \pm\infty$), but we can also use it to discuss their local behavior (as $x \to a$, where a is a finite value). Figure 1 summarizes arrow notation.

Symbol	Meaning
$x \to a^-$	x approaches a from the left ($x < a$ but close to a)
$x \to a^+$	x approaches a from the right ($x > a$ but close to a)
$x \to \infty$	x approaches infinity (x increases without bound)
$x \to -\infty$	x approaches negative infinity (x decreases without bound)
$f(x) \to \infty$	the output approaches infinity (the output increases without bound)
$f(x) \to -\infty$	the output approaches negative infinity (the output decreases without bound)
$f(x) \to a$	the output approaches a

Figure 1. Arrow Notation

Note that when a sign is included to the right of a value, it indicates the direction of approach. A "-" sign indicates approaching the value from the left, while a "+" sign indicates approaching it from the right. The direction is important because $f(x)$ may approach different values from the left and the right.

Asymptotes

An **asymptote** is a line that a graph approaches. There are three types of asymptotes: vertical, horizontal, and slant asymptotes. Asymptotes are often graphed as dotted lines. See Figure 2 and Figure 3 for some asymptote examples.

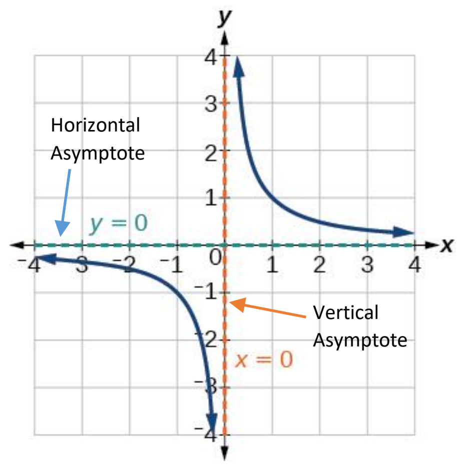

Figure 2. Vertical and Horizontal Asymptotes
Source: Adapted from [1]

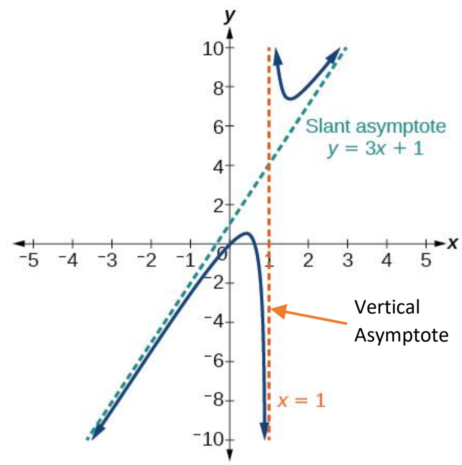

Figure 3. Vertical and Slant Asymptotes
Source: Adapted from [1]

Finding Vertical Asymptotes and Removable Discontinuities

More formally, a **vertical asymptote** is "a vertical line $x = a$ where the graph tends toward positive or negative infinity as the inputs approach a" [1]. Vertical asymptotes are part of a rational function's local behavior. Vertical asymptotes are only located where the input value causes division by zero. This makes sense because a graph is undefined when division by zero occurs. However, some input values that cause division by zero do *not* cause a vertical asymptote but instead cause a **removable discontinuity** (a hole in the graph marked by an open circle). Here is a strategy for determining the vertical asymptotes and removable discontinuities of a rational function:

1. Factor the numerator and denominator.
2. Set each factor in the denominator equal to zero and solve to find the denominator's zeros.
3. If a factor appears in both the numerator and the denominator, a removable discontinuity occurs at the corresponding zero found in step 2.
4. If a factor only appears in the denominator, a vertical asymptote occurs at the zero from step 2.

Finding Horizontal and Slant Asymptotes

A **horizontal asymptote** is "a horizontal line $y = b$ where the graph approaches the line as the inputs increase or decrease without bound" [1]. Horizontal (and slant) asymptotes therefore correspond to a rational function's end behavior, which is determined by the ratio of the leading terms of the numerator and denominator. There are three cases that can occur for a horizontal asymptote based on the degrees of the numerator and denominator polynomials. See Figure 4.

HORIZONTAL ASYMPTOTES OF RATIONAL FUNCTIONS

The horizontal asymptote of a rational function can be determined by looking at the degrees of the numerator and denominator.

- Degree of numerator *is less than* degree of denominator: horizontal asymptote at $y = 0$.
- Degree of numerator *is greater than degree of denominator by one*: no horizontal asymptote; slant asymptote.
- Degree of numerator *is equal to* degree of denominator: horizontal asymptote at ratio of leading coefficients.

Figure 4. Finding Horizontal Asymptotes
Source: [1]

A **slant asymptote** is defined by a linear equation. If you determine that a rational function has a slant asymptote using the test in Figure 4, you can find this linear equation using polynomial division. (You can review polynomial division in the week 6 resource.) Divide the rational function's numerator by its denominator. The quotient (without the remainder) is the equation of the slant asymptote!

Domains of Rational Functions

Recall that the domain of a polynomial is all real numbers. Therefore, the domain of a rational function, a ratio of polynomials, is all real numbers *except* those that cause the denominator polynomial to be equal to zero. In other words, we must exclude vertical asymptotes and removable discontinuities from the domain; the function is not defined at these locations.

Graphs of Rational Functions

Graphing a rational function is similar to graphing a polynomial, but the asymptotes must also be taken into account. See Figure 5 for the steps to graph a rational function.

HOW TO

Given a rational function, sketch a graph.

1. Evaluate the function at 0 to find the y-intercept.
2. Factor the numerator and denominator.
3. For factors in the numerator not common to the denominator, determine where each factor of the numerator is zero to find the x-intercepts.
4. Find the multiplicities of the x-intercepts to determine the behavior of the graph at those points.
5. For factors in the denominator, note the multiplicities of the zeros to determine the local behavior. For those factors not common to the numerator, find the vertical asymptotes by setting those factors equal to zero and then solve.
6. For factors in the denominator common to factors in the numerator, find the removable discontinuities by setting those factors equal to 0 and then solve.
7. Compare the degrees of the numerator and the denominator to determine the horizontal or slant asymptotes.
8. Sketch the graph.

Figure 5. Graphing a Rational Function
Source: [1]

To determine the local behavior around a vertical asymptote (step 5), look at the multiplicity of the corresponding factor in the denominator.

- If the factor has an odd multiplicity, $f(x) \to \infty$ on one side of the asymptote and $f(x) \to -\infty$ on the other side.
- If the factor has an even multiplicity, $f(x) \to \infty \text{ or } f(x) \to -\infty$ on both sides of the asymptote.

Reminders

Thank you for reading this resource! Here are a few reminders about other Precalculus opportunities:

- Group Tutoring for Precalculus is on Tuesdays from 5:00 to 6:00 p.m. through Microsoft Teams. If you have not received an email inviting you to sign up, and you want to attend, please email me at Sydney_Schirner1@baylor.edu, and I will make sure you are included!
- The Tutoring Center has great Precalculus videos on YouTube! You can find links for them at https://www.baylor.edu/support_programs/index.php?id=955694.
- If you want individual help, you can also schedule a FREE 30-minute appointment with one of our awesome Precalculus tutors at https://www.baylor.edu/tutoring.

References

[1] J. Abramson *et al*., *Precalculus*. Houston, Texas, USA: Rice Univ., 2014. [Online]. Available: https://openstax.org/details/books/precalculus