Instructions: Complete seven of the following problems. Start a new page for each problem.

1. Let $G$ be a finite group and let $p$ be a prime number.
   (a) How is the center of $G$ defined? $Z(G) =$
   (b) How is the centralizer of an element $x \in G$ defined? $C_G(x) =$
   (c) State and prove the class equation for $G$.
   (d) Show that if $G$ is a nontrivial $p$-group, then $Z(G)$ is nontrivial.

2. Let $G$ be a finite group whose order is divisible by the prime number $p$.
   (a) What is a Sylow $p$-subgroup of $G$?
   (b) State the three Sylow theorems for $G$.
   (c) What else can we say about the number of Sylow $p$-subgroups of $G$ if we assume that $G$ is a simple nonabelian group?
   (d) Show that there is no simple group of order 80.

3. Let $R$ be a commutative ring with $1 \neq 0$.
   (a) Complete the definition: An ideal $I$ of $R$ is a maximal ideal if .
   (b) Show that $R$ has at least one maximal ideal.
   (c) Show that an ideal $I$ of $R$ is a maximal ideal if and only if $R/I$ is a field.

4. Let $F$ be a field.
   (a) Briefly explain why the polynomial ring $F[x]$ is a PID.
   (b) Show that if $f(x) \in F[x]$ is a nonzero polynomial of degree $n$, then $f(x)$ has at most $n$ roots in $F$.
   (c) Show that if $F$ is finite, then the multiplicative group $F^\times = F - \{0\}$ is cyclic.

5. Let $F$ be a field and let $f(x) \in F[x]$ be a nonconstant polynomial.
   (a) Define what it means for $f(x)$ to be solvable by radicals over $F$. As part of your answer, also define what we mean by a radical field extension of $F$.
   (b) Define the Galois group of $f(x)$ and state Galois’ theorem.
   (c) Show that the polynomial $f(x) = x^5 - 6x^2 + 3$ is not solvable by radicals over $\mathbb{Q}$. You may take use (without proof) that $f(x)$ has exactly three distinct real roots and one pair of complex conjugate roots.
6. Let $A \in M_5(\mathbb{C})$ and suppose that the Smith normal form of $tI - A \in M_5(\mathbb{C}[t])$ is
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & t+1 & 0 & 0 \\
0 & 0 & 0 & (t+1)^2(t-2)^2 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]
(a) Find the minimal polynomial of $A$.
(b) Find the characteristic polynomial of $A$.
(c) Find the rational normal form of $A$.
(d) Find the Jordan normal form of $A$.
(e) View $V = \mathbb{C}^5$ as a $\mathbb{C}[t]$-module so that $tv = Av$ for all $v \in V$. Use the fundamental theorem for finitely generated modules over a PID to explain why $V$ is not a cyclic $\mathbb{C}[t]$-module.

7. Let $R$ be a commutative ring.

(a) Show that if $\pi : M \rightarrow N$ and $\sigma : N \rightarrow M$ are $R$-module homomorphisms such that $\pi \circ \sigma = \text{id}_N$, then $M = \ker \pi \oplus \text{im} \sigma$.

(b) Show that if $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is a short exact sequence of $R$-module homomorphisms and $N$ is a free $R$-module, then $M \cong L \oplus N$.

8. Let $R$ be a ring and let $L$ be a simple left $R$-module.

(a) Show that $\text{End}_R(L)$ is a division ring.

(b) Show that if $R$ is a $\mathbb{C}$-algebra and $\text{dim}_{\mathbb{C}}(L) < \infty$, then $\text{End}_R(L) \cong \mathbb{C}$.

(c) Suppose $R$ is a semisimple $\mathbb{C}$-algebra and $\text{dim}_{\mathbb{C}}(R) < \infty$. Use part (b) and the Wedderburn–Artin theorem to show that $R \cong M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$. 