Instructions: Choose 5 problems to solve and turn in your solutions via e-mail by May 25, 2021. This exam is open book, open notes. You are to work individually on this exam–absolutely no consulting with others.

[1] 10 points] Let \( f : [0, \infty) \to \mathbb{R} \) be Lebesgue integrable.

(a) Show that if \( f \) is uniformly continuous, then \( \lim_{x \to \infty} f(x) = 0 \).

(b) Is it sufficient for \( f \) to be continuous? Give a proof or a counterexample.
Let \((X, \mathcal{M}, \mu)\) be a measure space and let \(f\) be a nonnegative \(\mu\)-integrable function such that for each \(n \in \mathbb{N}\),
\[
\int f(x)^n \, d\mu = \int f(x) \, d\mu.
\]

(a) Show that the set \(G = \{x \in X : f(x) > 1\}\) has measure zero.

(b) Show that there exists some measurable set \(E \in \mathcal{M}\) such that \(f = \chi_E\) \(\mu\)-a.e.
[3] [10 points] Let \((X, \mathcal{B})\) be a measurable space and \(\langle \mu_n \rangle\) a sequence of finite measures on \((X, \mathcal{B})\) that converge setwise to a finite measure \(\mu\) (i.e. for each \(E \in \mathcal{B}\), \(\mu_n(E)\) converges to \(\mu(E)\)) and \(\langle f_n \rangle\) a sequence of nonnegative measurable functions on \(X\) that converge pointwise to the function \(f\).

Prove that
\[
\int f \, d\mu \leq \liminf \int f_n \, d\mu_n.
\]

Hint: Observe that if \(\phi \leq f\) is a simple function, then, for all \(\epsilon > 0\),
\[
X = \bigcup_{n \in \mathbb{N}} \{x : f_k(x) \geq (1 - \epsilon)\phi(x) \text{ for all } k \geq n\}.
\]
4 [10 points] Let $1 \leq p < \infty$, and let $L^p(\mathbb{R}^n)$ be endowed with its standard norm $\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx\right)^{1/p}$. Let $g$ be a measurable function on $\mathbb{R}^n$. Define $X_g$ to be the subspace of all $f \in L^p(\mathbb{R}^n)$ such that $fg \in L^p(\mathbb{R}^n)$, and endow $X_g$ with the same norm as $L^p$. Let $T : X_g \to L^p(\mathbb{R}^n)$ be given by $Tf = fg$.

(a) Show that $T$ is closed.

(b) Show that $T$ is bounded if and only if $g \in L^\infty(\mathbb{R}^n)$. 
[10 points] Let $f \in L^2(\mathbb{R})$. For each $k \in \mathbb{Z}$ define

$$f_k(x) = \int_k^{k+1} e^{2\pi i x \xi} \hat{f}(\xi) \, d\xi.$$  

Prove that

$$f = \sum_{k \in \mathbb{Z}} f_k,$$

where the sum converges in $L^2(\mathbb{R})$. (Hints: Prove that $\hat{\chi}_{[k,k+1)} \in L^1$ for each $k$ and apply the inverse Fourier transform. Use Exercise 60 from Chapter 5 in Folland.)
Let $H$ be a Hilbert space and let $\{x_n\}$ be a sequence that converges weakly to $x$ in $H$.

(a) Show that $\|x\| \leq \lim \inf \|x_n\|$. (Hint: $\langle x, x \rangle = \lim_{k \to \infty} \langle x_{n_k}, x \rangle$ for any subsequence $\{x_{n_k}\}$.)

(b) If $\|x_n\| \to \|x\|$, show that $x_n \to x$ strongly.
Let \((X, \mu)\) be a finite measure space. Let \(\{f_n\}\) be a sequence in \(L^p(X, \mu)\) and let \(f \in L^p(X, \mu)\). Assume \(f_n \to f\) a.e. and that \(\|f_n\|_p\) is bounded. Prove that \(\|f_n - f\|_q \to 0\) for all \(1 \leq q < p\). (Hint: use Egorov’s Theorem and H"older’s inequality.)