**Instructions:**

- This exam is closed book, closed notes.
- Write your solutions clearly and legibly in the space provided; no credit will be given for illegible solutions.
- Turn in scratch paper only if it contains part of your final answer.
- Show all work in your proofs; you will be graded on your reasoning.

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Complete the following statements of definitions and theorems.

(a) [1 point] If $X$ is a topological space, then the Borel $\sigma$-algebra on $X$ is...

(b) [1 point] Complete the statement of Fatou's Lemma: if $\{f_n\}$ is a sequence of non-negative measurable functions, then...

(c) [1 point] State the Monotone Convergence Theorem.

(d) [1 point] State the Dominated Convergence Theorem.

(e) [1 point] Complete the statement of the Lebesgue-Radon-Nikodym Theorem: if $\nu$ is a complex measure and $\mu$ is a $\sigma$-finite positive measure on $(X, \mathcal{M})$, then there exist...

(f) [1 point] Complete the statement of the Hahn-Banach Theorem: if $X$ is a real vector space, $p$ is a sublinear functional on $X$, $M$ is a subspace of $X$, and $f$ is a linear function on $M$ such that $f(x) \leq p(x)$ for all $x \in M$, then...
2 [10 points] Solve one of the following.

(a) Let $m$ be Lebesgue measure on $\mathbb{R}$. If $E$ is Lebesgue measurable and $0 < m(E) < \infty$, prove that for any $\alpha < 1$ there is an open interval $I$ such that $m(E \cap I) > \alpha m(I)$. (Hint: argue by contradiction.)

(b) If $\mu$ is a semifinite measure (every set of infinite measure contains a set of positive finite measure) and if $\mu(E) = \infty$, prove that for any $C > 0$ there exists $F \subseteq E$ with $C < \mu(F) < \infty$. (Hint: let $M = \sup\{\mu(F) : F \subseteq E, \mu(F) < \infty\}$. If $M < \infty$, first construct a set $F \subseteq E$ such that $\mu(F) = M$, then use this to derive a contradiction.)
3 [5 points] If \( \{f_n\} \) is a sequence of non-negative measurable functions such that \( f_n \) decreases pointwise to \( f \) and \( \int f_1 < \infty \), prove that \( \int f = \lim \int f_n \).

4 [5 points] Let \( \mu \) be a positive measure. A sequence of functions \( \{f_n\} \subset L^1(\mu) \) is called uniformly integrable if for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that whenever \( \mu(E) < \delta \), then \( \left| \int_E f_n \, d\mu \right| < \epsilon \) for all \( n \). Prove that if \( \{f_n\} \) converges to some \( f \) in \( L^1(\mu) \), then it is uniformly integrable.
(a) State the Closed Graph Theorem.

(b) Let $Y = C([0,1])$, the space of all continuous complex-valued functions on $[0,1]$, and let $X = C^1([0,1])$, the space of all continuously differentiable complex-valued functions on $[0,1]$, both equipped with the uniform norm $\|f\|_u = \sup_{x \in [0,1]} |f(x)|$.

Prove that the derivative map $\left(\frac{d}{dx}\right) : X \to Y$ is closed but not bounded. (To show it is closed, use facts from advanced calculus. To show it is not bounded, construct a sequence that is uniformly bounded such that the derivatives blow up.)

(c) Explain why part (b) does not contradict part (a).
[6] [10 points] Prove that every closed convex set $K$ in a Hilbert space has a unique element of minimal norm.
7 [10 points] Let $f \in L^2([0, 1])$. Prove that
\[ \lim_{n \in \mathbb{N}, n \to \infty} \int_0^1 f(x) \cos(2\pi nx) \, dx = 0. \]
[10 points] For \( f \in L^1(\mathbb{R}^n) \), we define the Fourier transform \( \hat{f} \) of \( f \) by

\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, dx.
\]

Also, let \( \Delta = \partial_1^2 + \cdots + \partial_n^2 \) denote the usual Laplacian.

Recall that \( e^{-\pi |x|^2} \) is its own Fourier transform.

(a) Define

\[
G_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t} \quad \forall t > 0, \ x \in \mathbb{R}^n.
\]

Then for all \( t > 0 \), show (i) \( \hat{G}_t \) is given by \( \hat{G}_t(\xi) = e^{-4\pi^2 \xi^2 t} \), (ii) \( \int G_t(x) \, dx = 1 \), and (iii) \( \partial_t \hat{G}_t(\xi) = \Delta \hat{G}_t(\xi) \).

(b) Let \( f \in L^p(\mathbb{R}^n) \) for some \( 1 \leq p < \infty \). Using part (a), sketch the proof showing that \( u(x,t) := f * G_t(x) \) (where \( * \) denotes convolution) satisfies the initial value problem

\[
\begin{cases}
\partial_t u(x,t) = \Delta u(x,t) & \text{if } x \in \mathbb{R}^n, t > 0 \\
u(x,0) = f(x) & \text{if } x \in \mathbb{R}^n
\end{cases}
\]

where the initial condition holds in the sense that \( \|u(\cdot,t) - f\|_p \to 0 \) as \( t \to 0 \).