1. Write careful definitions for the following.

   (a) Topology

   (b) Basis

   (c) Closed Set

   (d) Net convergence

   (e) Cluster point for a net

   (f) Continuous function
(g) Compact set

(h) Connected set

(i) Quotient space

(j) Closure of a set

(k) Path connected

(l) A *chain complex* is:

(m) A *short exact sequence* is:
(n) A simplicial complex is:

(o) Let $X$ be a topological space. For $p \geq 0$, the $p$-th singular chain group $S_p(X)$ and boundary $\partial : S_p(X) \to S_{p-1}(X)$ are given by:

(p) Let $\Lambda$ be an index set, and for each $\lambda \in \Lambda$, $X_\lambda$ a topological space. Let $X = \bigcup_{\lambda \in \Lambda} X_\lambda$. Then the topology on $X$ is coherent with the topologies on $X_\lambda$ if:

(q) A set $X$ in $\mathbb{R}^n$ is star convex with respect to a point $w$ if:

(r) A topological space $X$ is a CW complex if:

(s) Let $X$ be a CW complex. For $p \geq 0$, the $p$-th cellular chain group $S_p(X)$ and boundary $\partial : S_p(X) \to S_{p-1}(X)$ are given by:

(t) The projective $n$-space $P^n$ is defined to be:
2. Prove exactly ONE of the following theorems from class. You do not need to recopy the statement of the theorem.

   (a) Let \( X \) and \( Y \) be a topological spaces. A function \( f : X \to Y \) is continuous if, and only if, for every net \((x_\lambda)_{\lambda \in \Lambda}\) in \( X \) that converges to a point \( x \in X \) we have the net \((f(x_\lambda))_{\lambda \in \Lambda}\) converges to \( f(x) \in Y \).

   (b) Tychonoff’s Theorem

   (c) Let \( E \subset X \). If \( E \) is connected and \( E \subseteq A \subseteq \overline{E} \) then \( A \) is also connected.

3. Prove exactly ONE of the following theorems from class. You do not need to recopy the statement of the theorem.

   (a) Let \( X \) be compact. If \( E \) is a collection of closed sets with the FIP, then \( \cap E \) is non-empty.

   (b) Let \( X \) be a topological space. If each pair of points \( x,y \in X \) is in a set \( E_{x,y} \) that is connected, then \( X \) is connected.

4. Prove exactly ONE of the following theorems from class. You do not need to recopy the statement of the theorem.

   (a) (Zig-Zag Lemma) Let \( 0 \to C \xrightarrow{d} D \xrightarrow{\psi} E \to 0 \) be a short exact sequence of chain complexes. Prove that the long sequence of homology groups

   \[
   \cdots \to H_p(C) \xrightarrow{d_\ast} H_p(D) \xrightarrow{\psi_\ast} H_p(E) \xrightarrow{\partial_\ast} H_{p-1}(C) \to \cdots
   \]

   is exact. [You may assume that \( \partial_\ast \) is a well-defined homomorphism]

   (b) The generalized Jordan Curve Theorem Let \( n > 0 \) Let \( C \) be a subset of \( \mathbb{S}^n \) homeomorphic to then \( n-1 \) sphere. Then \( \mathbb{S}^n - C \) has precisely two components, of which \( C \) is the common topological boundary.

   (c) Zero-dimensional Homology Ket \( K \) be a simplical complex. Then the group \( H_0(K) \) is free abelian. If \( \{v_\alpha\} \) is a collection consisting of a single vertex from each component of \( |K| \), then the homology classes of the chains \( v_\alpha \) form a basis for \( H_0(K) \).

5. Complete TWO of the following problems. You must, of course, provide proofs (or counter examples) for your assertions.

   (a) Prove the Brouwer fixed-point theorem, i.e. prove that for \( n \geq 0 \) every continuous map from \( B^n \) to itself has a fixed point.

   (b) Let \( X \) be a subspace of \( \mathbb{R}^n \) which is star convex relative to the point \( w \). Then \( X \) is acyclic in singular homology.

   (c) State the Eilenberg-Steenrod Axioms for homology.

   (d) Let \( K,L \) be simplicial complexes and \( f,g : K \to L \) simplicial maps that are contiguous. Then there is a chain homotopy between \( f_\ast \) and \( g_\ast \), and hence \( f_\ast = g_\ast \).

   (e) Prove that \( \mathbb{R}^n \) is homeomorphic to \( \mathbb{R}^m \) if and only if \( n = m \).

6. Calculate the following

   (a) The homology groups of the Klein bottle, \( K \), and the connected sum of two Klein bottles, \( K \# K \), in all dimensions.

   (b) The fundamental group of the sphere, \( S^2 \).