1. Complete the following definitions (carefully)

(a) A basis for a topology on the set $X$ is a collection $\mathcal{B}$ of subsets of $X$ such that …

(b) The point $x$ is a limit point of the set $A$ in a topological space $X$ provided that …

(c) Let $X$ and $Y$ be topological spaces. A function $f : X \to Y$ is continuous if …

(d) Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. A basis for the product topology on $\prod_{\alpha \in J} X_\alpha$ is given by …

(e) Let $X$ be topological space. The component of $x$ in $X$ is …

(f) Let $X$ be a topological space. We say $X$ is Hausdorff if …

(g) Let $X$ and $Y$ be topological spaces. Two functions $f, f' : X \to Y$ are homotopic if …

(h) Let $X$ and $Y$ be a topological spaces and $f : X \to Y$ a continuous map. A set $U \subseteq Y$ is evenly covered by $f$ if …

(i) Let $X$ be a topological space and $A$ a subset of $X$. Then $A$ is a deformation retract of $X$ if …
2. Complete the following definitions (carefully)

(a) Let $X$ be a topological space. The (singular) homology group $H_k(X)$ is …

(b) Let $f : X \to Y$ be a continuous map. Then $f_* : H_i(X) \to H_i(Y)$ is given by …

(c) Let $(X, A)$ be a pair of topological space. The relative homology group $H_k(X, A)$ is …

(d) Let $(X, A)$ be a pair of topological space. Then $\partial : H_k(X, A) \to H_{k-1}(X)$ is given by …

(e) A pair $(X, A)$ of topological spaces is called a good pair if …

(f) Let $X$ be a CW complex. The cellular homology group $H^\text{CW}_i(X)$ is …

(g) Let $f : S^n \to S^n$ be a continuous map. Then the degree of $f$ is …

(h) Let $X$ be a topological space and let $G$ be an abelian group. The (singular) cohomology group $H^i(X; G)$ is …

(i) Let $f : X \to Y$ be a continuous map. Then $f^* : H^i(Y; G) \to H^i(Y; G)$ is given by …
3. Prove exactly ONE of the following theorems from class. You do not need to recopy the statement of the theorem.

(a) A nonempty subset $A \subseteq \mathbb{R}^n$ is compact if and only if $A$ is closed and bounded (in the standard metric on $\mathbb{R}^n$).

(b) The product of finitely many compact space is compact. NB: If you use any lemmas in this argument, you must also prove them.

(c) If $X$ is a compact Hausdorff space, then $X$ is a Baire space.

(d) Let $\pi : E \to B$ be a covering map with $\pi(e_0) = b_0$ and $\gamma : I \to B$ a path beginning at $b_0$. Then $\gamma$ lifts to a path $\tilde{\gamma} : I \to B$ beginning at $e_0$. (Nb: you do not need to prove the uniqueness of the lift.)
4. Complete **TWO** of the following problems.

(a) Let $A, B \subseteq X$, a topological space.
   i. Show that if $A$ is connected and $A \subseteq B \subseteq A$, then $B$ is also connected.
   ii. Prove or give a counterexample for each of the following equations: (1) $A \cup B = \overline{A \cup B}$
       and (2) $\overline{A \cap B} = \overline{A} \cap \overline{B}$.
   iii. Show that if $A$ and $B$ are connected, then $A \times B$ is connected.

(b) Suppose that $q : X \rightarrow Y$ is a quotient map. Prove that if $p^{-1}(y)$ is connected for each $y$ and $Y$ is connected, then so is $X$.

(c) Show that a compact Hausdorff space is normal.

(d) Show that a countable product of separable spaces is separable.
5. Complete **ALL** of the following problems.

   (a) Prove that the fundamental group of the circle is isomorphic to $\mathbb{Z}$.

   (b) Let $X$ be the complement of the $z$-axis in $\mathbb{R}^3$. Find $\pi_1(X)$ (provide an informal justification).

   (c) Let $Y$ be the space obtained by removing three distinct points from $\mathbb{R}^2$. Find $\pi_1(Y)$ (provide an informal justification).
6. Complete exactly TWO of the following problems.

(a) Let \( f, g : X \to Y \) be two homotopic continuous maps and let \( f_* \), \( g_* : H_k(X) \to H_k(Y) \) be the induced homomorphisms in homology. Sketch a proof that \( f_* = g_* \).

(b) State the snake lemma and explain how it is used to construct the long exact sequence in homology of a pair of topological spaces \((X, A)\).

(c) Let \( X \) be a CW complex with no two cells in adjacent dimensions. Prove that \( H_k^{CW}(X) \) is a free abelian group with a basis in one-to-one correspondence with the \( k \)-cells of \( X \).
7. Complete exactly **TWO** of the following problems.

(a) Use the long exact sequence of the good pair \((D^n, S^{n-1})\) to prove that

\[
H_k(S^n) \cong \begin{cases} 
\mathbb{Z} & \text{if } k = n \text{ or } k = 0; \\
0 & \text{otherwise.} 
\end{cases}
\]

*Remark:* You may use without proof that the quotient \(D^n / S^{n-1}\) is homeomorphic to \(S^n\).

(b) Calculate the local homology groups \(H_k(\mathbb{R}^n, \mathbb{R}^n - \{x\})\) and then use the result to prove that \(\mathbb{R}^n\) is homeomorphic to \(\mathbb{R}^m\) if and only if \(n = m\).

(c) Prove that the antipodal map \(f : S^n \to S^n, x \mapsto -x\), has degree \((-1)^{n+1}\).
8. Complete exactly **two** of the following problems.

(a) Let $K^2$ be the Klein bottle equipped with a $\Delta$-complex structure as shown below:

![Diagram of a Klein bottle]

Use the $\Delta$-complex structure to show that

$$H_k(K^2) \cong \begin{cases} 
\mathbb{Z} & \text{if } k = 0, \\
\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) & \text{if } k = 1, \\
0 & \text{if } k \geq 2.
\end{cases}$$

(b) Let $K^2$ be the Klein bottle as above. Use the universal coefficient theorem to compute the cohomology groups $H^k(K^2; G)$, where $G$ is an abelian group.

**Remark:** You may use without proof the following facts from homological algebra:

- $\text{Hom}(\_ , G)$ and $\text{Ext}^1(\_ , G)$ are additive functors
- $\text{Hom}(\mathbb{Z}, G) \cong G$
- $\text{Hom}(\mathbb{Z}/n\mathbb{Z}, G) \cong T_n(G) := \{ g \in G \mid ng = 0 \}$
- $\text{Ext}^1(\mathbb{Z}, G) = 0$
- $\text{Ext}^1(\mathbb{Z}/n\mathbb{Z}, G) \cong G/nG$

(c) Use a Mayer–Vietoris sequence to calculate the homology groups of a disk in the plane with two circular holes.