Gap probabilities for the Bures–Hall ensemble and the Cauchy–Laguerre two-matrix model

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With L. Wei, TTU

[Hall 1998]

What statistical ensemble corresponds to minimal prior knowledge about a quantum state?

1. What is a typical quantum state?
   - The setting for quantum mechanics
   - The Fubini-Study metric for projective Hilbert spaces
   - Density matrices
   - The Bures-Hall measure
   - The joint probability density function for eigenvalues of the density matrix
2. A statistic for the Bures-Hall ensemble: the gap probability
   - Pfaffian point processes
   - Cauchy-Laguerre Matrix model
   - Bi-orthogonal system of polynomials
   - Integrability and the dynamical system

- Complex Hermitian $n \times n$ matrix $\rho = \rho^\dagger$
- Unit trace $\mathrm{Tr}\rho = 1$
- Positive definite $\rho > 0$, i.e. $\forall \left|\psi\right\rangle$, $\left\langle\psi\right| \rho \left|\psi\right\rangle > 0$
- Pure state $\rho = \left|\psi\right\rangle\left\langle\psi\right|$ or $\rho^2 = \rho$
- Mixed state $\rho \neq \left|\psi\right\rangle\left\langle\psi\right|$
- Measured value of an observable, i.e. a Hermitian operator A, is $\mathrm{Tr}(\rho A)$
- Eigenvalues $0 \leq \rho_j \leq 1$, $\sum_{j=1}^{n} \rho_j = 1$
- Ex: Check that the Bloch Sphere for a single qubit $n = 2$

$$\rho(x, y, z) = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$$

satisfies all the requirements.

- Hilbert-Schmidt distance:

$$d^2_{HS}(\rho_A, \rho_B) = \mathrm{Tr}((\rho_A - \rho_B)^2)$$

Joint probability density function for the eigenvalues

$$P(\rho_1, \dots, \rho_n) = \frac{1}{C_n} \mathbb{1}_{\sum_{j=1}^n \rho_j = 1} \prod_{1 \le j < k \le n} \left(\rho_k - \rho_j\right)^2$$

- Bures-distance [Bures 1969]

$$d^2_B(\rho_A, \rho_B) = 2 - 2\mathrm{Tr}\sqrt{(\sqrt{\rho_A}\,\rho_B\,\sqrt{\rho_A})}$$

Ex: Joint probability density function for the eigenvalues [Hall 1998]

$$P(\rho_1, \dots, \rho_n) = \frac{1}{C_n} \mathbb{1}_{\sum_{j=1}^n \rho_j = 1} \prod_{j=1}^n \rho_j^{-1/2} \times \prod_{1 \le j < k \le n} \frac{\left(\rho_k - \rho_j\right)^2}{\left(\rho_k + \rho_j\right)}$$

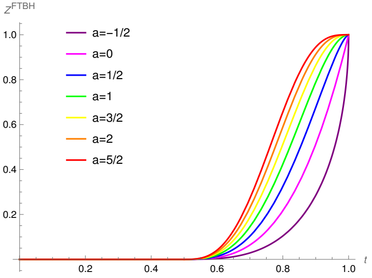
- Challenge Ex: Show

$$C_n = \frac{2^{-n(n-1)} \pi^{n/2}}{\Gamma(n^2/2)} \prod_{i=1}^n \Gamma(i+1)$$

Generalised gap probability for the Bures-Hall *fixed-trace* ensemble

$$Z^{\mathrm{FT}}(t;n,a,\xi)$$
$$:= \frac{1}{C^{\mathrm{FT}}(n,a,1)}\frac{1}{n!}\left(\int_0^\infty - \xi\int_t^\infty\right)d\rho_1 \dots \left(\int_0^\infty - \xi\int_t^\infty\right)d\rho_n$$
$$\times\,\delta\Big(\sum_{j=1}^n \rho_j - 1\Big)\prod_{j=1}^n \rho_j^a \prod_{1\le j<k\le n}\frac{(\rho_k-\rho_j)^2}{(\rho_k+\rho_j)}$$

Here $t > 0$, $\xi \in \mathbb{C}$, $\mathrm{Re}(a) > -1$.



Fixed-trace Bures-Hall gap probability $n = 2$, $\xi = 1$ case of $Z^{\mathrm{FT}}(t;2,a,1)$

Step 1 :

Generalised gap probability for the Bures-Hall *fixed-trace* ensemble

$$Z^{\mathrm{FT}}(t;n,a,r,\xi)$$

$$:= \frac{1}{C^{\mathrm{FT}}(n,a,1)}\frac{1}{n!}\left(\int_0^\infty - \xi\int_t^\infty\right)d\rho_1\ldots\left(\int_0^\infty - \xi\int_t^\infty\right)d\rho_n$$

$$\times\,\delta\Big(\sum_{j=1}^n \rho_j - r\Big)\prod_{j=1}^n \rho_j^a \prod_{1\le j<k\le n}\frac{(\rho_k-\rho_j)^2}{(\rho_k+\rho_j)}$$

Here $r > 0$, $t > 0$, $\xi \in \mathbb{C}$, $\mathrm{Re}(a) > -1$.

Step 2 :

Generalised gap probability for the *unconstrained* Bures-Hall ensemble

$$Z^{\mathrm{U}}(s;n,a,\xi)$$

$$:= \frac{1}{C^{\mathrm{U}}(n,a)}\frac{1}{n!}\left(\int_0^\infty - \xi\int_s^\infty\right)dx_1\ldots\left(\int_0^\infty - \xi\int_s^\infty\right)dx_n \prod_{1\le j<k\le n}\frac{(x_k-x_j)^2}{(x_k+x_j)}\prod_{j=1}^n x_j^a e^{-x_j}$$

Step 3 :
Generalised gap probability for the Cauchy-Laguerre Two Matrix Model

$$
\begin{aligned}
Z_n^{\mathrm{CL}}&(s,t;a,b;\xi,\psi) \\
&:= \frac{1}{C^{\mathrm{CL}}(n,a,b)}\frac{1}{(n!)^2}\left(\int_0^\infty - \xi\int_s^\infty\right)dx_1\ldots\left(\int_0^\infty - \xi\int_s^\infty\right)dx_n \\
&\quad \left(\int_0^\infty - \psi\int_t^\infty\right)dy_1\ldots\left(\int_0^\infty - \psi\int_t^\infty\right)dy_n \\
&\quad \times\prod_{j=1}^n x_j^a e^{-x_j} y_j^b e^{-y_j}\frac{\prod_{1\le j<k\le n}(x_k-x_j)(y_k-y_j)}{\prod_{1\le j,k\le n}(x_j+y_k)}\prod_{1\le j<k\le n}(x_k-x_j)(y_k-y_j)
\end{aligned}
$$

Step 4 :
The Unconstrained Bures-Hall generating function is related to the Cauchy-Laguerre Two Matrix Model by

$$
\left(Z^{\mathrm{U}}(s;n,a,\xi)\right)^2 = 2^n Z_n^{CL}(s,s;a,a+1,\xi,\xi)
$$

extending [Forrester+Kieburg 2016]

Bi-moments $M_{j,k}$ are defined by

$$M_{j,k} = \int_{S_1 \times S_2} dxdy \frac{w(x,y)}{x+y} x^j y^k < \infty, \quad \forall j,k \geq 0.$$

The Cauchy-Laguerre gap probability is a determinant of the truncated *Gram matrix*

$$Z_n^{\rm CL} = \det \left(M_{j,k} \right)_{j,k=0}^{n-1} \neq 0, \qquad n \geq 1$$

with

$$w(x,y) = w_1(x)\, w_2(y) = x^a e^{-x} y^b e^{-y}, \qquad \mathrm{Re}(a,b) > -1$$

Proposition (Bertola+Gekhtman+Szmigielski 2010)

Let two sequences of univariate moments be defined as

$$\alpha_j := \int_{S_1} dx\, w_1(x) x^j, \qquad \beta_j := \int_{S_2} dy\, w_2(y) y^j.$$

A key identity, which we call the Cauchy relation because it follows directly from the Cauchy kernel, for the bi-moments is the rank-1 condition

$$M_{j+1,k} + M_{j,k+1} = \alpha_j \beta_k.$$

Define an *inner product* over polynomial spaces $\cup_{n\geq 0}\Pi_n[x]$ using

$$\langle f,g\rangle := \int_{S_1\times S_2} dxdy\,\frac{w(x,y)}{x+y}f(x)g(y)$$

with $f,g \in \cup_{n\geq 0}\Pi_n[x]$
Two sequences of *bi-orthogonal polynomials* $\{P_n(x),Q_n(y)\}_{n=0}^{\infty}$ satisfying the orthogonality relation

$$\langle P_m,Q_n\rangle = \delta_{m,n}$$

Let two sequences of univariate moments be defined as

$$\pi_j := \int_{S_1} dx\, w_1(x)P_j, \qquad \eta_j := \int_{S_2} dy\, w_2(y)Q_j.$$

Proposition (BGS 2010)

Let the coefficients π_n, η_n be non-vanishing for all $n\geq 0$, and the norms h_n or Z_n^C similarly be non-vanishing. The bi-orthogonal polynomials $P_n(x)$, $Q_n(y)$ defined by the general orthogonality condition satisfy uncoupled, third-order scalar recurrence relations of the form

$$x\left(\frac{1}{\pi_{n+1}}P_{n+1}-\frac{1}{\pi_n}P_n\right) = r_{n,2}P_{n+2}+r_{n,1}P_{n+1}+r_{n,0}P_n+r_{n,-1}P_{n-1},$$

$$y\left(\frac{1}{\eta_{n+1}}Q_{n+1}-\frac{1}{\eta_n}Q_n\right) = s_{n,2}Q_{n+2}+s_{n,1}Q_{n+1}+s_{n,0}Q_n+s_{n,-1}Q_{n-1}.$$

Define the reproducing kernel

$$K_n^{0,0}(x,y) := \sum_{l=0}^{n} P_l(x)Q_l(y).$$

In [BGS 2010] the notion of a pair of *intertwinned* polynomials was introduced

$$\hat{P}_n(x) := -\sum_{k=0}^{n} \eta_k P_k(x),$$
$$\check{Q}_n(y) := -\sum_{j=0}^{n} \pi_j Q_j(y),$$

Proposition (WW 2022)

Let all the standard conditions apply. The Christoffel-Darboux sum for the reproducing kernel has the evaluation

$$(x+y)K_n^{0,0}(x,y) = \hat{P}_n(x)\check{Q}_n(y) + \frac{S_n}{S_{n+1}}\left[P_n(x)Q_{n+1}(y) + P_{n+1}(x)Q_n(y)\right].$$

Associated Functions of the first type of n-th order

$$P_n^{(1)}(z) := \int_{S_1} dx \frac{w_1(x)}{z-x} P_n(x), \qquad Q_n^{(1)}(z) := \int_{S_2} dy \frac{w_2(y)}{z-y} Q_n(y)$$

Associated Functions of the second type

$$P_n^{(2)}(z) := \int_{S_1 \times S_2} dxdy \frac{w(x,y)}{x+y} \frac{P_n(x)}{z-x}, \qquad Q_n^{(2)}(z) := \int_{S_1 \times S_2} dxdy \frac{w(x,y)}{x+y} \frac{Q_n(y)}{z-y}$$

3×3 matrix variables for $n \geq 0$

$$\mathcal{P}_n := \begin{pmatrix} P_{n+1}(x) & P_{n+1}^{(1)}(x) & P_{n+1}^{(2)}(x) \\ P_n(x) & P_n^{(1)}(x) & P_n^{(2)}(x) \\ P_{n-1}(x) & P_{n-1}^{(1)}(x) & P_{n-1}^{(2)}(x) \end{pmatrix}$$

Matrix system of first order recurrences with transfer matrices K_n and L_n

$$\mathcal{P}_{n+1} = K_n \mathcal{P}_n, \qquad \mathcal{Q}_{n+1} = L_n \mathcal{Q}_n$$

Proposition (WW 2022)

The kernels $K_n^{\mu,\nu}(x,y)$ for $\mu,\nu = 0,1,2$ are the bilinear forms

$$\pi_n \eta_n (x+y) K_n^{\mu,\nu}(x,y) = \mathcal{P}_n^{(\mu)}(x)^T G_n(x,y) \mathcal{Q}_n^{(\nu)}(y)$$

where $G(x,y)$ is the 3×3 matrix

$$G_n(x,y) = \begin{pmatrix} \frac{S_n^2}{S_{n+1}^2} & \frac{S_n}{S_{n+1}}\left[Y_{n,n} - y\right] & -\frac{S_{n-1}}{S_{n+1}} \\ \frac{S_n}{S_{n+1}}\left[X_{n,n} - x\right] & \left[X_{n,n} + y\right]\left[Y_{n,n} + x\right] & \frac{S_{n-1}}{S_n}\left[Y_{n,n} + x\right] \\ -\frac{S_{n-1}}{S_{n+1}} & \frac{S_{n-1}}{S_n}\left[X_{n,n} + y\right] & \frac{S_{n-1}^2}{S_n^2} \end{pmatrix}$$

Re-define matrix variables

$$\mathcal{P}_n := \begin{pmatrix} P_{n+1}(x) & e^x x^{-a} P^{(1)}_{n+1}(x) & x^{-a-b} P^{(2)}_{n+1}(x) \\ P_n(x) & e^x x^{-a} P^{(1)}_{n}(x) & x^{-a-b} P^{(2)}_{n}(x) \\ P_{n-1}(x) & e^x x^{-a} P^{(1)}_{n-1}(x) & x^{-a-b} P^{(2)}_{n-1}(x) \end{pmatrix}$$

Overdetermined system

$$\partial_x \mathcal{P}_n = \mathcal{A}_n \mathcal{P}_n, \qquad \partial_s \mathcal{P}_n = \mathcal{B}_n \mathcal{P}_n, \qquad \partial_t \mathcal{P}_n = \mathcal{C}_n \mathcal{P}_n$$

Partial-fraction form of the Coefficients

$$\mathcal{A}_n(x) = \frac{1}{x}\mathcal{A}_n^{(0)} + \frac{1}{x-s}\mathcal{A}_n^{(s)} + \frac{1}{x+t}\mathcal{A}_n^{(-t)} + \mathcal{A}_n^{(\infty)}$$

$$\mathcal{B}_n(x) = \frac{1}{x-s}\mathcal{B}_n^{(s)} + \mathcal{B}_n^{(\infty)}, \qquad \mathcal{C}_n(x) = \frac{1}{x+t}\mathcal{C}_n^{(-t)} + \mathcal{C}_n^{(\infty)}$$

Explicitly

$$\mathcal{A}_n(x) = \begin{pmatrix} 0 & \frac{\pi_{n+1}}{\pi_n} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{\pi_{n-1}}{\pi_n} & 0 \end{pmatrix} + \frac{1}{x}\mathcal{A}_n^{(0)}$$

$$-\frac{1}{x-s}\xi s^a e^{-s}\frac{1}{\pi_n\eta_n}\mathcal{P}_n^{(0)}(s)\otimes\Big(G_n(s,-s)\mathcal{Q}_n^{(1)}(-s)\Big)^T + \frac{1}{x+t}\psi t^b e^{-t}\frac{1}{\pi_n\eta_n}\mathcal{P}_n^{(1)}(-t)\otimes\Big(G_n(-t,t)\mathcal{Q}_n^{(0)}(t)\Big)^T$$

Deduce three *Schlesinger equations*

$$\partial_s \mathcal{A}_n - \partial_x \mathcal{B}_n = [\mathcal{B}_n, \mathcal{A}_n]\,, \qquad \partial_t \mathcal{A}_n - \partial_x \mathcal{C}_n = [\mathcal{C}_n, \mathcal{A}_n]\,, \qquad \partial_t \mathcal{B}_n - \partial_s \mathcal{C}_n = [\mathcal{C}_n, \mathcal{B}_n]$$

σ-functions

$$\sigma_n(s,t) := s\partial_s \log Z_n^{\mathrm{CL}}\,, \qquad \tau_n(s,t) := t\partial_t \log Z_n^{\mathrm{CL}}$$

The σ-functions have evaluations in terms of *reproducing kernels*

$$\sigma_n(s,t) = -\xi s^{a+1} e^{-s} K_{n-1}^{0,1}(s,-s), \qquad \tau_n(s,t) = -\psi t^{b+1} e^{-t} K_{n-1}^{1,0}(-t,t)$$

Kernels defined by

$$K_n^{0,1}(x,y) := \sum_{l=0}^{n} P_l(x) Q_l^{(1)}(y), \qquad K_n^{1,0}(x,y) := \sum_{l=0}^{n} P_l^{(1)}(x) Q_l(y)$$

The σ-forms for deformed Cauchy-Laguerre bi-orthogonal ensemble have the evaluations

$$\sigma_{n+1}(s,t) = s\mathrm{Tr}\left(\begin{pmatrix} 0 & \frac{\pi_{n+1}}{\pi_n} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{\pi_{n-1}}{\pi_n} + \frac{S_n}{S_{n-1}} & 0 \end{pmatrix} \mathcal{A}_n^{(s)}\right) + \mathrm{Tr}\left(\mathcal{A}_n^{(\Sigma)} \mathcal{A}_n^{(s)}\right) - \frac{t}{(s+t)} \mathrm{Tr}\left(\mathcal{A}_n^{(-t)} \mathcal{A}_n^{(s)}\right),$$

A number of statistics have been calculated for the Bures-Hall ensemble which are global in nature such as:

- Quantum purity $\mathbb{E}[\mathrm{Tr}(\rho^2)]$.
- von Neumann entropy $\mathbb{E}[-\mathrm{Tr}(\rho \log \rho)]$ and its higher moments.

However

- The gap probability is a refined local statistic of the density matrix spectrum, especially of the extreme eigenvalues.
- It is now accessible with the tools of "integrable probability" and Riemann-Hilbert methods.
- It is a rare example of an integrable formulation of a *Pfaffian point processes*, whereas virtually all the other known cases are of *determinantal point processes*.
- It is a rank 3 isomonodromic system and is beyond the Painlevé class which has rank 2.

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