Recent progress on exact moments of entanglement entropies

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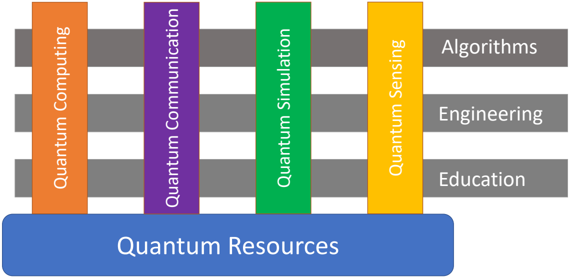
Quantum Information Theory

- **Quantum information theory:** theoretical underpinnings of quantum sciences that enable quantum technologies
- **Quantum technologies:** quantum computing, quantum communication, quantum simulation, and quantum sensing



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- The soul of bipartite model is the operation of partial trace (purification) of ρ leading to a **reduced density matrix**

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- **Fermionic Gaussian (FG) ensemble** equips quadratic random Hamiltonians with anticommuting properties. Relevant to quantum circuits complexity

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- Entanglement spectrum

$$f_{\mathrm{HS}}\left(\boldsymbol{\lambda}\right) \propto \delta\left(1 - \sum_{i=1}^{m} \lambda_i\right) \prod_{1 \le i < j \le m} \left(\lambda_i - \lambda_j\right)^2 \prod_{i=1}^{m} \lambda_i^{n-m}$$

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$$\mathcal{D} = \left\{ 0 \le \lambda_m < \ldots < \lambda_1 \le 1, \quad \sum_{i=1}^{m} \lambda_i = 1 \right\}$$

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  - Maximally-entangled state: $\lambda_1 = \lambda_2 = \ldots \lambda_m = 1/m$

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- Quantum purity (experimentally observable[*†])

$$P = \mathrm{tr}\left(\rho_A^2\right) = \sum_{i=1}^{m} \lambda_i^2$$

[*] Islam et al. [2015] Measuring entanglement entropy in a quantum many-body system, *Nature*

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$$C = \mathrm{tr}\left(\rho_A \ln^2 \rho_A\right) - \mathrm{tr}^2\left(\rho_A \ln \rho_A\right)$$

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- Degree of entanglement is encoded in the moments of entropies and capacity

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von Neumann Entropy over Hilbert-Schmidt Ensemble

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- **Mean value**: conjectured by Page'93*, proved by Sánchez'95† (among other proofs)

$$\kappa_1 = \psi_0(mn + 1) - \psi_0(n) - \frac{m + 1}{2n}$$

where

$$\psi_0(l) = -\gamma + \sum_{k=1}^{l-1} \frac{1}{k}, \qquad \gamma \approx 0.5772$$

*Page [1993] Average entropy of a subsystem, *Phys. Rev. Lett.*

†Sánchez [1995] Simple proof of Page's conjecture on the average entropy of a subsystem, *Phys. Rev. E*

von Neumann Entropy over HS Ensemble

- **Variance:** conjectured by Vivo-Pato-Oshanin'16*, proved by Wei'17†

$$\kappa_2 = -\psi_1(mn+1) + \frac{m+n}{mn+1}\psi_1(n) - \frac{(m+1)(m+2n+1)}{4n^2(mn+1)}$$

where

$$\psi_1(l) = \frac{\pi^2}{6} - \sum_{k=1}^{l-1}\frac{1}{k^2}$$

*Vivo, Pato, Oshanin [2016] Random pure states: Quantifying bipartite entanglement beyond the linear statistics, *Phys. Rev. E*

†Wei [2017] Proof of Vivo-Pato-Oshanin's conjecture on the fluctuation of von Neumann entropy, *Phys. Rev. E*

von Neumann Entropy over HS Ensemble

- **Skewness:** conjectured by Bianchi-Donà'19*, proved by Wei'20†

$$\kappa_3 = \psi_2(mn+1) - \frac{m^2+3mn+n^2+1}{(mn+1)(mn+2)}\psi_2(n) + \frac{(m^2-1)(mn-3n^2+1)}{n(mn+1)^2(mn+2)}\psi_1(n) + \frac{(m+1)(2+4m+2m^2+6n+8mn+4m^2n+2m^3n+10n^2+3mn^2+3m^2n^2+4mn^3)}{4n^3(mn+1)^2(mn+2)}$$

where

$$\psi_2(l) = -2\zeta(3) + 2\sum_{k=1}^{l-1}\frac{1}{k^3}, \qquad \zeta(3) \approx 1.20206$$

*Bianchi-Donà [2019] Typical entanglement entropy in the presence of a center: Page curve and its variance, *Phys. Rev. D*

†Wei [2020] Skewness of von Neumann entanglement entropy, *J. Phys. A*

von Neumann Entropy over HS Ensemble

- **Kurtosis:** by Huang-Wei-Collaku'21*

*Huang-Wei-Collaku [2021] Kurtosis of von Neumann entanglement entropy, *J. Phys. A*

von Neumann Entropy over HS Ensemble

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Summary of the first four cumulants:

$$\kappa_1 = a_1\psi_0(mn+1) + a_2\psi_0(n) + a_3$$
$$\kappa_2 = b_1\psi_1(mn+1) + b_2\psi_1(n) + b_3$$
$$\kappa_3 = c_1\psi_2(mn+1) + c_2\psi_2(n) + c_3\psi_1(n) + c_4$$
$$\kappa_4 = d_1\psi_3(mn+1) + d_2\psi_3(n) + d_3\psi_2(n) + d_4\psi_1^2(n) + d_5\psi_1(n) + d_6$$

*Huang-Wei-Collaku [2021] Kurtosis of von Neumann entanglement entropy, *J. Phys. A*

Results over Bures-Hall Ensemble

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- **First moments of purity and von Neumann entropies:** conjectured by Sarkar-Kumar'19*, proved by Wei'20†

$$\kappa_1 = \psi_0\left(mn - \frac{m^2}{2} + 1\right) - \psi_0\left(n + \frac{1}{2}\right)$$

*Sarkar-Kumar [2019] Bures-Hall ensemble: spectral densities and average entropies, *J. Phys. A*

†Wei [2020] Proof of Sarkar-Kumar's conjectures on average entanglement entropies over the Bures-Hall ensemble, *J. Phys. A*

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- **Second moment of Neumann entropy:** by Wei'20‡

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†Wei [2020] Proof of Sarkar-Kumar's conjectures on average entanglement entropies over the Bures-Hall ensemble, *J. Phys. A*

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- **Second and third moments of purity:** by Li-Wei'21§

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†Wei [2020] Proof of Sarkar-Kumar's conjectures on average entanglement entropies over the Bures-Hall ensemble, *J. Phys. A*

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Arbitrary number of particles:

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- **Average entanglement capacity:** Huang-Wei (work in progress)

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$$\kappa_2 = I_A - I_B$$

$$\begin{aligned} I_A \;=\;& a_1 + a_2\psi_0(n) + a_3\psi_0(n-m) + a_4\left(\psi_0(n) - \psi_0(m) + \psi_0(1)\right) \times \\ & \psi_0(n-m) + a_5\left(\psi_0^2(n-m) - \psi_1(n-m)\right) + \\ & a_6\sum_{k=1}^{m}\frac{\psi_0(k+n-m)}{k} \end{aligned}$$

$$\begin{aligned} I_B \;=\;& b_1 + b_2\psi_0(n) + b_3\psi_0(n-m) + b_4\psi_0^2(n) + b_5(\psi_0(n) - \psi_0(m) + \\ & \psi_0(1))\psi_0(n-m) + b_6\left(\psi_0^2(n-m) + \psi_1(n) - \psi_1(n-m)\right) + \\ & b_7\sum_{k=1}^{m}\frac{\psi_0(k+n-m)}{k} \end{aligned}$$

Example: unsimplifiable bases in the kurtosis calculation (HS ensemble)

$$\Omega_1 = \sum_{k=1}^{m} \frac{\psi_0(k+n-m)}{k} \qquad \Omega_2 = \sum_{k=1}^{m} \frac{\psi_0^2(k+n-m)}{k}$$

$$\Omega_3 = \sum_{k=1}^{m} \frac{\psi_1(k+n-m)}{k} \qquad \Omega_4 = \sum_{k=1}^{m} \frac{\psi_0(k+n-m)}{k^2}$$

$$\Omega_5 = \sum_{k=1}^{m} \frac{\psi_0^2(k+n-m)}{k^2} \qquad \Omega_6 = \sum_{k=1}^{m} \frac{\psi_0^2(k+n-m)}{k+n-m}$$

$$\Omega_7 = \sum_{k=1}^{m} \frac{\psi_0(k+n-m)\psi_0(k)}{k} \qquad \Omega_8 = \sum_{k=1}^{m} \frac{\psi_0^3(k+n-m)}{k}$$

$$\Omega_9 = \sum_{k=1}^{m} \frac{\psi_0^2(k+n-m)\psi_0(k)}{k} \qquad \Omega_{10} = \sum_{k=1}^{m} \frac{\psi_0^3(k+n-m)}{k+n-m}$$

$$\Omega_{11} = \sum_{k=1}^{m} \frac{\psi_1(k+n-m)}{k+n-m} \qquad \Omega_{12} = \sum_{k=1}^{m} \frac{\psi_0(k+n-m)\psi_1(k+n-m)}{k}$$

$$\Omega_{13} = \sum_{k=1}^{m} \frac{\psi_1(k+n-m)\psi_0(k)}{k} \qquad \Omega_{14} = \sum_{k=1}^{m} \frac{\psi_0(k+n-m)\psi_1(k+n-m)}{k+n-m}$$

$$\Omega_{15} = \sum_{k=1}^{m} \frac{\psi_2(k+n-m)}{k} \qquad \Omega_{16} = \sum_{k=1}^{m} \frac{\psi_2(k+n-m)}{k+n-m}$$