Solvability of some integro-differential equations with drift and superdiffusion

Messoud Efendiev, (Helmholtz Zentrum Munchen)

Vitali Vougalter, (University of Toronto)

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Existence of stationary solutions of nonlocal reaction- diffusion equations: existence of biological species.

1. Introduction.

Integro-differential equations: nonlocal consumption of resources, intra-specific competition. Nonlocal interaction of neurons.

$$-\sqrt{-\frac{d^2}{dx^2}}u + b\frac{du}{dx} + au + \int_{-\infty}^{\infty} G(x-y)F(u(y),y)dy = 0 \qquad (1)$$

with constants $a \geq 0$, $b \in \mathbb{R}$, $b \neq 0$. Appear in cell population dynamics. Spatial variable x corresponds to the cell genotype. $u(x,t)$ denotes the cell density as a function of their genotype and time. The evolution of cell density is due to cell proliferation, mutations and drift. The superdiffusion term: $\sqrt{-\frac{d^2}{dx^2}}u = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |p|\widehat{u}(p)e^{ipx}dp$. The function $F(u,x)$ is the rate of cell birth, depends on u and x (density dependent proliferation). $G(x-y)$ is the proportion of newly born cells which change their genotype from y to x. It depends on the distance between the genotypes.

N. Apreutesei, N. Bessonov, V. Volpert, V. V., Discrete Contin. Dyn. Syst., Ser. B (2010).

H. Berestycki, G. Nadin, B. Perthame, L. Ryzhik, Nonlinearity (2009).

S. Genieys, V. Volpert, P. Auger, Math. Model. Nat. Phenom (2006).

The existence of stationary solutions of (1) via the fixed point technique.

V. V., V. Volpert, J. Pseudo-Differ. Oper. Appl. (2018).

V. V., V. Volpert, Rend. Semin. Mat. Univ. Padova (2017).

The sequence of iterated equations

$$-\sqrt{-\frac{d^2}{dx^2}}u_m + b\frac{du_m}{dx} + au_m + \int_{-\infty}^{\infty} G_m(x-y)F(u_m(y), y)dy = 0, \quad (2)$$

with constants $a \geq 0$, $b \in \mathbb{R}$, $b \neq 0$ and $m \in \mathbb{N}$. $G_m(x) \to G(x)$ as $m \to \infty$ in the appropriate function spaces.

2. The Whole Real Line Case.

Sobolev space:

$$\|u\|^2_{H^2(\mathbb{R})} = \|u\|^2_{L^2(\mathbb{R})} + \left\| \frac{d^2u}{dx^2} \right\|^2_{L^2(\mathbb{R})}.$$

No drift term:

$$\sqrt{-\frac{d^2}{dx^2}} - a : H^1(\mathbb{R}) \to L^2(\mathbb{R}), \quad a \geq 0$$

self-adjoint, non Fredholm operator. Its essential spectrum

$$\sigma_{ess} = [-a, +\infty).$$

$0 \in \sigma_{ess}$, no bounded inverse. The obstacle to solve our equation.

V.V., V. Volpert, J. Pseudo-Differ. Oper. Appl. (2018).

Auxiliary problem

$$\sqrt{-\frac{d^2}{dx^2}}u - b\frac{du}{dx} - au = \int_{-\infty}^{\infty} G(x-y)F(v(y),y)dy. \tag{3}$$

The non-selfadjoint operator

$$L_{a,b} := \sqrt{-\frac{d^2}{dx^2}} - b\frac{d}{dx} - a : \ H^1(\mathbb{R}) \to L^2(\mathbb{R}). \tag{4}$$

Standard Fourier transform:

$$\widehat{G}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(x)e^{-ipx}dx, \quad p \in \mathbb{R}. \tag{5}$$

The essential spectrum of $L_{a,b}$:

$$\lambda_{a,b}(p) = |p| - a - ibp, \quad p \in \mathbb{R}.$$

$a > 0 :$ $L_{a,b}$ is Fredholm, $\sigma_{ess}(L_{a,b})$ away from the origin.

$a = 0 :$ $L_{a,b}$ is non Fredholm, $\sigma_{ess}(L_{a,b})$ contains the origin.

For the convolution: $\widehat{f*g}(p) = \sqrt{2\pi}\widehat{f}(p)\widehat{g}(p)$. Upper bounds

$$\|\widehat{G}(p)\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2\pi}}\|G(x)\|_{L^1(\mathbb{R})}, \quad \|p\widehat{G}(p)\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2\pi}}\left\|\frac{dG(x)}{dx}\right\|_{L^1(\mathbb{R})}.$$

Elliott H. Lieb, M. Loss, Analysis (1997).

Sobolev norm using Fourier transform (5)

$$\|u\|^2_{H^2(\mathbb{R})} = \|\widehat{u}(p)\|^2_{L^2(\mathbb{R})} + \|p^2\widehat{u}(p)\|^2_{L^2(\mathbb{R})}. \tag{6}$$

Uniqueness of solutions of the auxiliary equation.

Suppose some $v \in H^2(\mathbb{R})$, two solutions $u_{1,2} \in H^2(\mathbb{R})$ of (3). Their difference $w(x) := u_1(x) - u_2(x) \in H^2(\mathbb{R})$ solves

$$\sqrt{-\frac{d^2}{dx^2}}w - b\frac{dw}{dx} - aw = 0.$$

$L_{a,b} : H^1(\mathbb{R}) \to L^2(\mathbb{R})$: no nontrivial zero modes, $w(x) \equiv 0$ on $\mathbb{R}$.

Let $v(x) \in H^2(\mathbb{R})$ be arbitrary. Apply Fourier transform (5) to both sides of equation (3):

$$\widehat{u}(p) = \sqrt{2\pi}\frac{\widehat{G}(p)\widehat{f}(p)}{|p| - a - ibp}, \quad p^2\widehat{u}(p) = \sqrt{2\pi}\frac{p^2\widehat{G}(p)\widehat{f}(p)}{|p| - a - ibp} \tag{7}$$

where

$$\widehat{F}(v(x), x)(p) = \widehat{f}(p).$$

Auxiliary quantities for $G(x) \in W^{1,1}(\mathbb{R})$

$$N_{a,\ b} := max\left\{\left\|\frac{\widehat{G}(p)}{|p| - a - ibp}\right\|_{L^\infty(\mathbb{R})}, \quad \left\|\frac{p^2\widehat{G}(p)}{|p| - a - ibp}\right\|_{L^\infty(\mathbb{R})}\right\}$$

with constants $a \geq 0,\ b \in \mathbb{R},\ b \neq 0$.

If $a > 0,\ b \in \mathbb{R},\ b \neq 0$ then $N_{a,\ b} < \infty$, no orthogonality conditions needed.

If $a = 0$, $b \in \mathbb{R}$, $b \neq 0$ and in addition $xG(x) \in L^1(\mathbb{R})$ then $N_{0,\ b} < \infty$ if and only if

$$(G(x), 1)_{L^2(\mathbb{R})} = 0 \tag{8}$$

holds. From (5):

$$\widehat{G}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(x)dx = \frac{1}{\sqrt{2\pi}} (G(x), 1)_{L^2(\mathbb{R})} = 0.$$

$$\frac{d\widehat{G}(p)}{dp} = \frac{-i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xG(x)e^{-ipx}dx, \quad xG(x) \in L^1(\mathbb{R}).$$

No drift term. V.V., V.Volpert, J. Pseudo-Differ. Oper. Appl. (2018).

Orthogonality relations

$$\left(G(x), \frac{e^{\pm iax}}{\sqrt{2\pi}}\right)_{L^2(\mathbb{R})} = 0, \quad a > 0.$$

$$(G(x), 1)_{L^2(\mathbb{R})} = 0, \quad a = 0.$$

More regular problem with drift.

Fixed point argument

$$|\widehat{u}(p)| \le \sqrt{2\pi} N_{a,\ b} |\widehat{f}(p)|, \quad |p^2 \widehat{u}(p)| \le \sqrt{2\pi} N_{a,\ b} |\widehat{f}(p)|.$$

Use (6) to estimate the norm as

$$\|u\|^2_{H^2(\mathbb{R})} \le 4\pi N^2_{a,\ b} \|F(v(x), x)\|^2_{L^2(\mathbb{R})} < \infty.$$

It is assumed that

$$|F(v, x)| \le k|v| + h(x), \quad v \in \mathbb{R}, \quad x \in \mathbb{R}, \quad k > 0$$

with

$$h(x) : \mathbb{R} \to \mathbb{R}^+, \quad h(x) \in L^2(\mathbb{R}).$$

For an arbitrary $v(x) \in H^2(\mathbb{R})$ there exists a unique solution $u(x) \in H^2(\mathbb{R})$ of (3) with its Fourier image given by (7).

The map $T_{a,b} : H^2(\mathbb{R}) \to H^2(\mathbb{R})$ is well defined.

To show: $T_{a,b} : H^2(\mathbb{R}) \to H^2(\mathbb{R})$ is a strict contraction.

Choose arbitrarily $v_{1,2}(x) \in H^2(\mathbb{R})$. Their images under our map $u_{1,2} = T_{a,b}v_{1,2} \in H^2(\mathbb{R})$, namely

$$\sqrt{-\frac{d^2}{dx^2}}u_1 - b\frac{du_1}{dx} - au_1 = \int_{-\infty}^{\infty} G(x-y)F(v_1(y), y)dy,$$

$$\sqrt{-\frac{d^2}{dx^2}}u_2 - b\frac{du_2}{dx} - au_2 = \int_{-\infty}^{\infty} G(x-y)F(v_2(y), y)dy.$$

Estimate

$$|\widehat{u}_1(p) - \widehat{u}_2(p)| \leq \sqrt{2\pi}N_{a,\ b}|\widehat{f}_1(p) - \widehat{f}_2(p)|,$$

$$|p^2\widehat{u}_1(p) - p^2\widehat{u}_2(p)| \leq \sqrt{2\pi}N_{a,\ b}|\widehat{f}_1(p) - \widehat{f}_2(p)|$$

with $\widehat{F}(v_{1,2}(x), x)(p) = \widehat{f}_{1,2}(p)$. For the norm

$$\|u_1 - u_2\|^2_{H^2(\mathbb{R})} \leq 4\pi N^2_{a,\ b}\|F(v_1(x), x) - F(v_2(x), x)\|^2_{L^2(\mathbb{R})}.$$

Sobolev embedding: $v_{1,2}(x) \in H^2(\mathbb{R}) \subset L^\infty(\mathbb{R})$.

Assume the Lipschitz continuity with a constant $l > 0$

$$|F(v_1, x) - F(v_2, x)| \le l|v_1 - v_2|, \quad any \quad v_{1,2} \in \mathbb{R}, \quad x \in \mathbb{R}.$$

We obtain

$$\|T_{a,b}v_1 - T_{a,b}v_2\|_{H^2(\mathbb{R})} \le 2\sqrt{\pi}N_{a,\ b}l\|v_1 - v_2\|_{H^2(\mathbb{R})}$$

with the constant $2\sqrt{\pi}N_{a,\ b}l < 1$ as assumed.

The Fixed Point Theorem.

Unique $v_{a,b} \in H^2(\mathbb{R})$, such that $T_{a,b}v_{a,b} = v_{a,b}$, which is the only solution of equation (1) in $H^2(\mathbb{R})$.

Suppose $v_{a,b} \equiv 0$. Assumed $supp\widehat{F}(0, x) \cap supp\widehat{G}$ is a set of nonzero Lebesgue measure in $\mathbb{R}$. Contradiction. Therefore, $v_{a,b}$ is nontrivial.

The existence in the sense of sequences of the solution on $\mathbb{R}$.

For $m \in \mathbb{N}$ the iterated kernels $G_m(x) \in W^{1,1}(\mathbb{R})$, $G_m(x) \to G(x)$ in $W^{1,1}(\mathbb{R})$ as $m \to \infty$.

If $a=0$, assume $xG_m(x)\in L^1(\mathbb{R}),\ xG_m(x)\to xG(x)$ in $L^1(\mathbb{R})$ as $m\to\infty$, the orthogonality conditions

$$(G_m(x),1)_{L^2(\mathbb{R})}=0,\quad m\in\mathbb{N} \tag{9}$$

hold. Auxiliary iterated expressions with $m\in\mathbb{N}$

$$N_{a,\ b,\ m}:=max\left\{\left\|\frac{\widehat{G}_m(p)}{|p|-a-ibp}\right\|_{L^\infty(\mathbb{R})},\quad \left\|\frac{p^2\widehat{G}_m(p)}{|p|-a-ibp}\right\|_{L^\infty(\mathbb{R})}\right\}$$

with constants $a\geq 0,\ b\in\mathbb{R},\ b\neq 0$. Assume that

$$2\sqrt{\pi}N_{a,\ b,\ m}l\leq 1-\varepsilon \tag{10}$$

all $m\in\mathbb{N}$, some fixed $0<\varepsilon<1$. Evidently

$$\|\widehat{G}_m(p)-\widehat{G}(p)\|_{L^\infty(\mathbb{R})}\leq\frac{1}{\sqrt{2\pi}}\|G_m(x)-G(x)\|_{L^1(\mathbb{R})}\to 0,\quad m\to\infty \tag{11}$$

as assumed.

Limiting argument when $a = 0$.

$$|(G(x), 1)_{L^2(\mathbb{R})}| = |(G(x) - G_m(x), 1)_{L^2(\mathbb{R})}| \leq \|G_m(x) - G(x)\|_{L^1(\mathbb{R})} \to 0$$

as $m \to \infty$ as assumed. Thus,

$$(G(x), 1)_{L^2(\mathbb{R})} = 0. \tag{12}$$

We have

$$\frac{\widehat{G}_m(p)}{|p| - a - ibp} \to \frac{\widehat{G}(p)}{|p| - a - ibp}, \quad m \to \infty,$$

$$\frac{p^2 \widehat{G}_m(p)}{|p| - a - ibp} \to \frac{p^2 \widehat{G}(p)}{|p| - a - ibp}, \quad m \to \infty,$$

in $L^\infty(\mathbb{R})$, such that via the triangle inequality

$$\left\| \frac{\widehat{G}_m(p)}{|p| - a - ibp} \right\|_{L^\infty(\mathbb{R})} \to \left\| \frac{\widehat{G}(p)}{|p| - a - ibp} \right\|_{L^\infty(\mathbb{R})}, \quad m \to \infty,$$

$$\left\| \frac{p^2 \widehat{G}_m(p)}{|p| - a - ibp} \right\|_{L^\infty(\mathbb{R})} \to \left\| \frac{p^2 \widehat{G}(p)}{|p| - a - ibp} \right\|_{L^\infty(\mathbb{R})}, \quad m \to \infty.$$

Furthermore,

$$2\sqrt{\pi} N_{a,\ b} l \leq 1 - \varepsilon.$$

Each equation (2) admits a unique solution $u_m(x) \in H^2(\mathbb{R}),\ m \in \mathbb{N}$. Limiting equation (1) has a unique solution $u(x) \in H^2(\mathbb{R})$ similarly.

Using the Fourier transform (5) we arrive at

$$\widehat{u}(p) = \sqrt{2\pi} \frac{\widehat{G}(p) \widehat{\varphi}(p)}{|p| - a - ibp}, \quad \widehat{u}_m(p) = \sqrt{2\pi} \frac{\widehat{G}_m(p) \widehat{\varphi}_m(p)}{|p| - a - ibp}, \quad m \in \mathbb{N},$$

where $\widehat{\varphi}(p)$ and $\widehat{\varphi}_m(p)$ denote the Fourier images of $F(u(x), x)$ and $F(u_m(x), x)$ under (5) respectively. We proved that

$$u_m(x) \to u(x) \quad in \quad H^2(\mathbb{R}) \quad as \quad m \to \infty.$$

The Problem on the Finite Interval.

$$-\sqrt{-\frac{d^2}{dx^2}}u + b\frac{du}{dx} + au + \int_0^{2\pi} G(x-y)F(u(y),y)dy = 0 \qquad (13)$$

with constants $a \geq 0,\ b \in \mathbb{R},\ b \neq 0$ on $I := [0, 2\pi]$.

Related sequence of approximate equations with $m \in \mathbb{N}$

$$-\sqrt{-\frac{d^2}{dx^2}}u_m + b\frac{du_m}{dx} + au_m + \int_0^{2\pi} G_m(x-y)F(u_m(y),y)dy = 0. \qquad (14)$$

Function space when $a > 0$

$$H^2(I) = \{u(x) : I \to \mathbb{R} \mid u(x), u''(x) \in L^2(I),\ u(0) = u(2\pi),\ u'(0) = u'(2\pi)\}.$$

Auxiliary constrained subspace when $a = 0$

$$H_0^2(I) = \{u(x) \in H^2(I) \mid (u(x), 1)_{L^2(I)} = 0\}. \qquad (15)$$

The Fredholm operator when $a = 0$

$$\sqrt{-\frac{d^2}{dx^2}} - b\frac{d}{dx} : \quad H_0^1(I) \to L^2(I)$$

has the trivial kernel. $H_0^2(I)$ is a Hilbert space as well.

P.D. Hislop, I.M. Sigal, Introduction to spectral theory with applications to Schrödinger operators (1996).

$$\mathcal{L}_{a,b} := \sqrt{-\frac{d^2}{dx^2}} - b\frac{d}{dx} - a : \quad H^1(I) \to L^2(I), \quad a > 0 \tag{16}$$

is Fredholm, non-selfadjoint, its set of eigenvalues

$$\lambda_{a,b}(n) = |n| - a - ibn, \quad n \in \mathbb{Z}.$$

Its eigenfunctions are the standard Fourier harmonics $\frac{e^{inx}}{\sqrt{2\pi}}$, $n \in \mathbb{Z}$.

V.V., V.Volpert, J. Pseudo-Differ. Oper. Appl. (2018)

Solvability of equation (13) without a drift.

Auxiliary problem

$$\sqrt{-\frac{d^2}{dx^2}}u - b\frac{du}{dx} - au = \int_0^{2\pi} G(x-y)F(v(y),y)dy. \qquad (17)$$

Uniqueness of solutions for the auxiliary problem.

Suppose for some $v(x) \in H^2(I)$ there are two solutions $u_{1,2}(x) \in H^2(I)$ of (17). The difference $w(x) := u_1(x) - u_2(x) \in H^2(I)$ solves

$$\sqrt{-\frac{d^2}{dx^2}}w - b\frac{dw}{dx} - aw = 0.$$

For $\mathcal{L}_{a,b} : H^1(I) \to L^2(I)$ no nontrivial zero modes, $a > 0$. Therefore, $w(x)$ vanishes in I.

Fourier transform on the finite interval.

$G(x) : I \to \mathbb{R},\ G(0) = G(2\pi).$

$$G_n := \int_0^{2\pi} G(x)\frac{e^{-inx}}{\sqrt{2\pi}}dx, \quad n \in \mathbb{Z} \tag{18}$$

and

$$G(x) = \sum_{n=-\infty}^{\infty} G_n \frac{e^{inx}}{\sqrt{2\pi}}.$$

Clearly, $\|G_n\|_{l^\infty} \leq \frac{1}{\sqrt{2\pi}}\|G(x)\|_{L^1(I)}, \quad \|nG_n\|_{l^\infty} \leq \frac{1}{\sqrt{2\pi}}\left\|\frac{dG(x)}{dx}\right\|_{L^1(I)}.$

Sobolev norm using Fourier transform (18)

$$\|u\|^2_{H^2(I)} = \sum_{n=-\infty}^{\infty} |u_n|^2 + \sum_{n=-\infty}^{\infty} |n^2 u_n|^2. \tag{19}$$

Let $v(x) \in H^2(I)$ be arbitrary. Apply Fourier transform (18) to (17).

$$u_n = \sqrt{2\pi}\frac{G_n f_n}{|n| - a - ibn}, \quad n^2 u_n = \sqrt{2\pi}\frac{n^2 G_n f_n}{|n| - a - ibn}, \quad n \in \mathbb{Z}, \tag{20}$$

where $f_n := F(v(x), x)_n$.

Auxiliary expressions for $G(x) \in C(I),\ \dfrac{dG(x)}{dx} \in L^1(I),\ G(0) = G(2\pi)$

$$\mathcal{N}_{a,\ b} := max\left\{\left\|\frac{G_n}{|n| - a - ibn}\right\|_{l^\infty}, \quad \left\|\frac{n^2 G_n}{|n| - a - ibn}\right\|_{l^\infty}\right\}$$

with constants $a \geq 0,\ b \in \mathbb{R},\ b \neq 0$.

If $a > 0,\ b \in \mathbb{R},\ b \neq 0$ then $\mathcal{N}_{a,\ b} < \infty$, no orthogonality conditions.

If $a = 0,\ b \in \mathbb{R},\ b \neq 0$ then $\mathcal{N}_{0,\ b} < \infty$ if and only if

$$(G(x), 1)_{L^2(I)} = 0 \tag{21}$$

holds. From (18) we have

$$G_0 = \frac{1}{\sqrt{2\pi}}(G(x), 1)_{L^2(I)} = 0.$$

Fixed point argument.

$$|u_n| \leq \sqrt{2\pi}\mathcal{N}_{a,\ b}|f_n|, \quad |n^2 u_n| \leq \sqrt{2\pi}\mathcal{N}_{a,\ b}|f_n|.$$

Use (19) to estimate the norm as

$$\|u\|^2_{H^2(I)} \leq 4\pi\mathcal{N}^2_{a,\ b}\|F(v(x),x)\|^2_{L^2(I)} < \infty.$$

We assume that

$$|F(v,x)| \leq k|v| + h(x), \quad v \in \mathbb{R}, \quad x \in \mathbb{I}, \quad k > 0,$$

where

$$h(x) : I \to \mathbb{R}^+, \quad h(x) \in L^2(I).$$

For an arbitrary $v(x) \in H^2(I)$ there exists a unique $u(x) \in H^2(I)$ satisfying (17) with its Fourier image given by (20).

The map $\tau_{a,b} : H^2(I) \to H^2(I)$ is well defined, $a > 0$.

To show: $\tau_{a,b} : H^2(I) \to H^2(I)$ is a strict contraction.

Consider any $v_{1,2}(x) \in H^2(I)$. Their images under our map $u_{1,2} = \tau_{a,b}v_{1,2} \in H^2(I)$, such that

$$\sqrt{-\frac{d^2}{dx^2}}u_1 - b\frac{du_1}{dx} - au_1 = \int_0^{2\pi} G(x-y)F(v_1(y),y)dy,$$

$$\sqrt{-\frac{d^2}{dx^2}}u_2 - b\frac{du_2}{dx} - au_2 = \int_0^{2\pi} G(x-y)F(v_2(y),y)dy.$$

Estimate for $n \in \mathbb{Z}$

$$|u_{1,n} - u_{2,n}| \le \sqrt{2\pi}\mathcal{N}_{a,\ b}|f_{1,n} - f_{2,n}|,$$

$$|n^2(u_{1,n} - u_{2,n})| \le \sqrt{2\pi}\mathcal{N}_{a,\ b}|f_{1,n} - f_{2,n}|,$$

where $F(v_j(x),x)_n = f_{j,n},\ j = 1,2.$ For the norm

$$\|u_1 - u_2\|^2_{H^2(I)} \le 4\pi\mathcal{N}^2_{a,\ b}\|F(v_1(x),x) - F(v_2(x),x)\|^2_{L^2(I)}.$$

Sobolev embedding: $v_{1,2}(x) \in H^2(I) \subset L^\infty(I).$ Assume the Lipschitz continuity with a constant $l > 0$

$$|F(v_1,x) - F(v_2,x)| \le l|v_1 - v_2|, \quad any \quad v_{1,2} \in \mathbb{R}, \quad x \in I.$$

We arrive at

$$\|\tau_{a,b}v_1 - \tau_{a,b}v_2\|_{H^2(I)} \le 2\sqrt{\pi}\mathcal{N}_{a,\ b}l\|v_1 - v_2\|_{H^2(I)}$$

with the constant $2\sqrt{\pi}\mathcal{N}_{a,\ b}l < 1$ as assumed.

The Fixed Point Theorem.

Unique $v_{a,b} \in H^2(I)$, such that $\tau_{a,b}v_{a,b} = v_{a,b}$, which is the only solution of equation (13) in $H^2(I)$.

Suppose $v_{a,b} \equiv 0$ in I. Assumed $G_n F(0,x)_n \neq 0$ for some $n \in \mathbb{Z}$. Contradiction. Thus, $v_{a,b}$ is nontrivial.

The existence in the sense of sequences of the solution on I.

$$G_m(x) \in C(I),\ \frac{dG_m(x)}{dx} \in L^1(I),\ G_m(x) \to G(x)\ in\ C(I),$$
$$\frac{dG_m(x)}{dx} \to \frac{dG(x)}{dx}\ in\ L^1(I)\ as\ m \to \infty,\ G_m(0) = G_m(2\pi).$$

If $a = 0$, assume that the orthogonality conditions $(G_m(x), 1)_{L^2(I)} = 0, \quad m \in \mathbb{N}$ hold. Therefore, $G_{m,0} = 0,\ m \in \mathbb{N}$.

Auxiliary iterated expressions with $m \in \mathbb{N}$

$$\mathcal{N}_{a,\ b,\ m} := max\left\{\left\|\frac{G_{m,n}}{|n|-a-ibn}\right\|_{l^\infty}, \quad \left\|\frac{n^2 G_{m,n}}{|n|-a-ibn}\right\|_{l^\infty}\right\}$$

with constants $a \ge 0,\ b \in \mathbb{R},\ b \neq 0.$ Assume that

$$2\sqrt{\pi}\mathcal{N}_{a,\ b,\ m} l \le 1-\varepsilon \tag{22}$$

all $m \in \mathbb{N}$, some fixed $0 < \varepsilon < 1.$ Upper bound

$$\|G_n\|_{l^\infty} \le \frac{1}{\sqrt{2\pi}}\|G(x)\|_{L^1(I)}, \tag{23}$$

such that

$$\|G_{m,n}-G_n\|_{l^\infty} \le \frac{1}{\sqrt{2\pi}}\|G_m-G\|_{L^1(I)} \le \sqrt{2\pi}\|G_m-G\|_{C(I)} \to 0, \quad m \to \infty$$

as assumed.

Limiting argument when $a = 0$.

$$|(G(x), 1)_{L^2(I)}| = |(G(x) - G_m(x), 1)_{L^2(I)}| \leq 2\pi \|G_m(x) - G(x)\|_{C(I)} \to 0$$

as $m \to \infty$ as assumed. Thus,

$$(G(x), 1)_{L^2(I)} = 0, \tag{24}$$

equivalently $G_0 = 0$. We have

$$\frac{G_{m,n}}{|n| - a - ibn} \to \frac{G_n}{|n| - a - ibn}, \quad m \to \infty$$

$$\frac{n^2 G_{m,n}}{|n| - a - ibn} \to \frac{n^2 G_n}{|n| - a - ibn}, \quad m \to \infty$$

in l^∞, such that via the triangle inequality

$$\left\| \frac{G_{m,n}}{|n| - a - ibn} \right\|_{l^\infty} \to \left\| \frac{G_n}{|n| - a - ibn} \right\|_{l^\infty}, \quad m \to \infty,$$

$$\left\| \frac{n^2 G_{m,n}}{|n| - a - ibn} \right\|_{l^\infty} \to \left\| \frac{n^2 G_n}{|n| - a - ibn} \right\|_{l^\infty}, \quad m \to \infty.$$

Furthermore, $2\sqrt{\pi}\mathcal{N}_{a,\ b}l \le 1 - \varepsilon$. Each equation (14) admits a unique solution $u_m(x),\ m \in \mathbb{N}$ in $H^2(I),\ a > 0$ and in $H_0^2(I),\ a = 0$, argument above. Limiting equation (13) has a unique solution $u(x)$ in $H^2(I),\ a > 0$ and in $H_0^2(I),\ a = 0$ similarly.

Using the Fourier transform (18) we obtain

$$u_n = \sqrt{2\pi}\frac{G_n \varphi_n}{|n| - a - ibn}, \quad u_{m,n} = \sqrt{2\pi}\frac{G_{m,n}\varphi_{m,n}}{|n| - a - ibn}, \quad n \in \mathbb{Z}, \quad m \in \mathbb{N},$$

where φ_n and $\varphi_{m,n}$ stand for the Fourier images of $F(u(x), x)$ and $F(u_m(x), x)$ under (18) respectively. We proved that $u_m(x) \to u(x)$ as $m \to \infty$ in $H^2(I),\ a > 0$ and in $H_0^2(I),\ a = 0$. Generalized to systems of equations. M. Efendiev, V.V., preprint (2022).