Chain sequences and Zeros of a perturbed R_{II} type recurrence relation

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Baylor Fest 2022
Baylor University, Texas USA

May 24, 2022

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Preliminaries

☞ Applying Gram-Schmidt process on $\{1, x, x^2, x^3, ...\}$, we obtain the sequence of polynomials which satisfies [1]

$$\int \mathcal{P}_n(x)\mathcal{P}_m(x)d\mu = \delta_{nm},$$

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- Orthogonal polynomials on real line (OPRL): TTRR

$$\mathcal{R}_{n+1}(x) = (x - b_n)\mathcal{R}_n(x) - \gamma_n\mathcal{R}_{n-1}(x),$$
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- Orthogonal polynomials on unit circle (OPUC): Szegő recurrence [2]

$$\begin{bmatrix}\phi_{n+1}(z)\\ \phi_{n+1}^*(z)\end{bmatrix} = \begin{bmatrix} z & -\overline{\alpha_n}\\ -\alpha_n z & 1\end{bmatrix}\begin{bmatrix}\phi_n(z)\\ \phi_n^*(z)\end{bmatrix},$$

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R_I and R_{II} polynomials

☞ R_I type recurrence relation and R_I polynomials [3]

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- R_{II} type recurrence relation and R_{II} polynomials

$$\mathcal{P}_{n+1}(x) = \rho_n(x - c_n)\mathcal{P}_n(x) - \lambda_n(x - a_n)(x - b_n)\mathcal{P}_{n-1}(x), \; n \geq 0,$$
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- Orthogonality: $\exists$ rational function $\psi_n(x) = \dfrac{\mathcal{P}_n(x)}{\prod_{j=1}^{n}(x - a_j)(x - b_j)}$

$$\mathfrak{N}\left[x^k \psi_n(x)\right] \neq 0, \quad 0 \leq k < n,$$

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☞ Polynomials of second kind

$$\mathcal{Q}_{n+1}(x) = \rho_n(x - c_n)\mathcal{Q}_n(x) - \lambda_n(x - a_n)(x - b_n)\mathcal{Q}_{n-1}(x), \; n \geq 1,$$

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$$\mathcal{P}_{n+1} = (x - c_{n+1})\mathcal{P}_n - \lambda_{n+1}(x^2 + 1)\mathcal{P}_{n-1}, \quad n \geq 1,$$

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$\{\lambda_n\}_{n\geq 1}$ - positive chain sequence

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✍ The transformation

$$\xi(x) = \frac{x + i}{x - i},$$

yields

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Connection with unit circle

- Sequence of para-orthogonal polynomials $\{r_n\}$

$$r_n(\xi) = \frac{2^n \mathcal{P}_n(x)}{(x-i)^n}, \quad n \geq 1,$$
$$r_{n+1}(\xi) = ((1+ic_{n+1})z + (1-ic_{n+1}))r_n(\xi) - 4\lambda_{n+1}\xi r_{n-1}(\xi),$$

- Related OPUC

$$\phi_{n-1}(\xi) = \frac{r_n(\xi) - 2(1-l_n)r_{n-1}(\xi)}{(z-1)\prod_{j=1}^{n}(1+ic_j)}, \quad n \geq 1,$$

OPUC directly from R_{II} polynomials

$$\phi_{n-1}(\xi) = \frac{-i2^{n-1}}{\prod_{j=1}^{n}(1+ic_j)} \frac{1}{(x-i)^{n-1}}[\mathcal{P}_n(x) - (1-l_n)(x-i)\mathcal{P}_{n-1}(x)], \quad n \geq 1,$$

Motivation

$$\mathcal{P}_{n+1} = (x - c_{n+1})\mathcal{P}_n - \lambda_{n+1}(x^2+1)\mathcal{P}_{n-1}, \quad n \geq 1, \quad \mathcal{P}_0 = 1, \quad \mathcal{P}_1 = x - c_1,$$

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☞ Perturbations

$$c_n^* = c_n + \mu_k \delta_{n,k}, \qquad \tilde{\lambda}_n = \nu_k^{\delta_{n,k}} \lambda_n$$

$$(\text{co} - \text{recursive}), \qquad (\text{co} - \text{dilated})$$

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- Co-recursive OPRL [5]
- Co-dilated OPRL [6]
- Co-modified OPRL [7]

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- Perturbation in recurrence coefficients of TTRR :
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Motivation. . .

- Perturbation in Verblunsky coefficients of Szegő recurrence : Co-polynomials on unit circle (COPUC) [9]
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- Perturbations of recurrence coefficients in the higher order recurrences and its extensions to Sobolev OPRL [11] [12]

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13 Vinay Shukla and A. Swaminathan, Rational spectral transformation of continued fractions associated to a perturbed R_I type recurrence relations, arXiv: 2201.05422, 22 pages, 2022.

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Perturbed R_{II} polynomials

- Perturbed R_{II} type recurrence relation

$$\tilde{\mathcal{P}}_{n+1} = (x - c_n^*)\tilde{\mathcal{P}}_n - \tilde{\lambda}_n(x^2+1)\tilde{\mathcal{P}}_{n-1},$$

- Generalised co-modified R_{II} polynomials $\mathcal{P}_n(x;\mu_k,\nu_k) \equiv \tilde{\mathcal{P}}_n$

$$\begin{aligned}
\tilde{\mathcal{P}}_{n+1} &= (x - c_n)\tilde{\mathcal{P}}_n - \lambda_n(x^2+1)\tilde{\mathcal{P}}_{n-1}, && n < k,\\
\tilde{\mathcal{P}}_{k+1} &= (x - c_k - \mu_k)\tilde{\mathcal{P}}_k - \nu_k\lambda_k(x^2+1)\tilde{\mathcal{P}}_{k-1}, && n = k,\\
\tilde{\mathcal{P}}_{n+1} &= (x - c_n)\tilde{\mathcal{P}}_n - \lambda_n(x^2+1)\tilde{\mathcal{P}}_{n-1}, && n \geq k+1.
\end{aligned}$$

- Co-polynomials of R_{II} type [14]

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- Co-polynomials of R_{II} type [14]
- Generalised co-recursive and co-dilated R_{II} polynomials

$$\nu_k = 1 \quad \textit{and} \quad \mu_k = 0.$$

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Theorem (V. Shukla and AS)

Let $\{\gamma_n\}_{n\geq 1}$ be the sequence of Verblunsky coefficients of the corresponding COPUC associated to

$$\mathcal{P}_{n+1}(x) = (x - a_{n+1})\mathcal{P}_n(x) - b_{n+1}(x^2+1)\mathcal{P}_{n-1}(x), \quad n \geq 0,$$

$$a_{n+1} = \begin{cases} c_{n+1}, & n \neq k \\ a_{k+1}, & n = k \end{cases}, \qquad b_{n+1} = \begin{cases} \lambda_{n+1}, & n \neq k \\ b_{k+1}, & n = k \end{cases},$$

Then for $n \geq k$,

$$\gamma_{n-1} = \frac{1 - ia_n}{1 + ia_n}\frac{1 + ic_n}{1 - ic_n}\frac{1 - ic_{n+1}}{1 - ia_{n+1}} \left[\alpha_{n-1} - \frac{1}{\tau_n}\left\{\frac{2(l_{n+1} - l'_{n+1}) + i(c_{n+1} - a_{n+1})}{1 - ic_{n+1}}\right\}\right],$$

where $\{l'_{n+1}\}_{n\geq 0}$ is the minimal parameter sequence of $\{b_{n+1}\}_{n\geq 1}$.

Example

- $c_n = 0$, $n \geq 1$, $\lambda_2 = 1/2$ and $\lambda_{n+1} = 1/4$, $n \geq 2$

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- $\{\lambda_{n+1}\}_{n\geq 1}$ is a SPPCS with $l_1 = 0$ and $\{l_{n+1}\}_{n\geq 1} = 1/2$

$$\begin{aligned}
\mathcal{P}_n(x) &= \left(\frac{x-i}{2}\right)^n + \left(\frac{x+i}{2}\right)^n, \quad n \geq 1, \\
r_n(z) &= z^n + 1, \quad n \geq 1, \\
\phi_n(z) &= z^n, \quad n \geq 0.
\end{aligned}$$

Co-dilation and Chain sequences

Example

- ☞ $c_n = 0$, $n \geq 1$, $\lambda_2 = 1/2$ and $\lambda_{n+1} = 1/4$, $n \geq 2$
- ☞ $\{\lambda_{n+1}\}_{n\geq 1}$ is a SPPCS with $l_1 = 0$ and $\{l_{n+1}\}_{n\geq 1} = 1/2$

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$$r_n(z) = z^n + 1, \quad n \geq 1,$$
$$\phi_n(z) = z^n, \quad n \geq 0.$$

- ☞ Verblunsky coefficients are $\alpha_n = 0$, $n \geq 0$ and corresponding probability measure μ is the Lebesgue measure given by $\boxed{d\mu(z) = \frac{1}{2i\pi z}dz.}$

Recovering new polynomials and the measure

- Co-dilated sequence $\{\lambda'_{n+1} = 1/4\}_{n\geq 1}$ i.e. $\lambda'_2 = \nu_2\lambda_2$ where $\nu_2 = 1/2$

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- **Consequences:** Not a SPPCS

$$l'_{n+1} = \frac{n}{2n+2}, \quad n \geq 0, \quad \text{and} \quad \mathrm{M}'_{\mathrm{n+1}} = \frac{1}{2}, \quad \mathrm{n} \geq 0,$$

- Perturbed Verblunsky coefficients $\gamma_{n-1} = -\dfrac{1}{n+1}$

Recovering new polynomials and the measure

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- Perturbed Verblunsky coefficients $\gamma_{n-1} = -\dfrac{1}{n+1}$
- Monic OPUC can be constructed

$$\phi'_n(z) = \frac{(n+1)z^n + nz^{n-1} + (n-1)z^{n-2} + \ldots + 2z + 1}{n+1}, \quad n \geq 0.,$$

$$r'_n(z) = \frac{z^{n+1} - 1}{z - 1}, \quad n \geq 0,$$

$$\mathcal{P}'_n(x) = i\left(\frac{x-i}{2}\right)^{n+1} - i\left(\frac{x+i}{2}\right)^{n+1}, \quad n \geq 0.$$

- Measure of orthogonality is found to be $\boxed{d\mu(z) = \dfrac{(1-z)(z-1)}{4i\pi z^2}dz}$

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Distribution of Zeros

Let X be the set of zeros of $\mathcal{P}_{k-1}(x)$.

Theorem (V. Shukla and AS)

For $x \in \mathbb{R} \backslash X$, the following formulas hold:

$$\mathcal{P}_n(x; \mu_k, \nu_k) = \mathcal{P}_n(x), \qquad n < k,$$

$$\mathcal{P}_n(x; \mu_k, \nu_k) = \mathcal{P}_n(x) - \mathcal{S}_k(x)\mathcal{P}^{(k)}_{n-k}(x), \qquad n \geq k,$$

where $\mathcal{S}_k(x) = \mu_k \mathcal{P}_k(x) + (\nu_k - 1)\lambda_k(x^2 + 1)\mathcal{P}_{k-1}(x)$.

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where $\mathcal{S}_k(x) = \mu_k\mathcal{P}_k(x) + (\nu_k - 1)\lambda_k(x^2+1)\mathcal{P}_{k-1}(x)$.

Corollary

The co-recursive polynomials $\mathcal{P}_n(x;\mu_k)$ and $\mathcal{P}_n(x)$ share at most the zeros of $\mathcal{P}_k$.

Theorem (V. Shukla and AS)

Let $n \geq k$ and $x_j^{(n)}(\mu)$ and $x_j^{(n)}$, $j = 1, 2, \ldots l$, be the l non common real zeros corresponding to $\mathcal{P}_n(x; \mu_k)$ and $\mathcal{P}_n(x)$.

- *If $\mu < 0$, then*

$$x_l^{(n)}(\mu) < x_l^{(n)} < x_{l-1}^{(n)}(\mu) < x_{l-1}^{(n)} < \ldots < x_1^{(n)}(\mu) < x_1^{(n)}.$$

- *If $\mu > 0$, then the role of the zeros $x_j^{(n)}(\mu)$ and $x_j^{(n)}$, $j = 1, 2, \ldots l$, gets interchanged.*

✍ **Case 1:** when $\mathcal{P}_n(x)$ and $\mathcal{P}_n(x;\mu_k)$ have common zeros.

$$\mathcal{P}_{n+1} = x\mathcal{P}_n - \frac{1}{4}(x^2+1)\mathcal{P}_{n-1}, \quad n \geq 1.$$

Example

For $\mu_k < 0$, $\mathcal{P}_9(x)$ and $\mathcal{P}_9(x;-0.7)$ have four zeros in common .

Figure: Zeros of $\mathcal{P}_9(x)$ (red diamonds) and $\mathcal{P}_9(x;-0.7)$ (blue circles)

✷ Other zeros interlace

- When $\mathcal{P}_n(x)$ and $\mathcal{P}_n(x;\mu_k)$ have no common zeros :
- Complementary Routh-Romanowski polynomials

$$\mathcal{P}_{n+1} = \left(x - \frac{\theta}{\zeta + n - 1}\right)\mathcal{P}_n - \frac{1}{4}\frac{n(2\zeta + n - 1)}{(\zeta + n - 1)(\zeta + n)}(x^2 + 1)\mathcal{P}_{n-1},$$

- When $\mathcal{P}_n(x)$ and $\mathcal{P}_n(x;\mu_k)$ have no common zeros :
- Complementary Routh-Romanowski polynomials

$$\mathcal{P}_{n+1} = \left(x - \frac{\theta}{\zeta+n-1}\right)\mathcal{P}_n - \frac{1}{4}\frac{n(2\zeta+n-1)}{(\zeta+n-1)(\zeta+n)}(x^2+1)\mathcal{P}_{n-1},$$

Example

$\mu_k > 0$, $\mathcal{P}_7(x)$ and $\mathcal{P}_7(x;1.2)$ have no common zeros

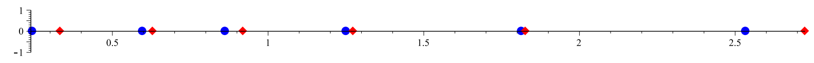

Figure: Zeros of $\mathcal{P}_6(x)$ (red diamonds) and $\mathcal{P}_6(x;-0.3)$ (blue circles)

- When $\mathcal{P}_n(x)$ and $\mathcal{P}_n(x;\mu_k)$ have no common zeros :
- Complementary Routh-Romanowski polynomials

$$\mathcal{P}_{n+1} = \left(x - \frac{\theta}{\zeta+n-1}\right)\mathcal{P}_n - \frac{1}{4}\frac{n(2\zeta+n-1)}{(\zeta+n-1)(\zeta+n)}(x^2+1)\mathcal{P}_{n-1},$$

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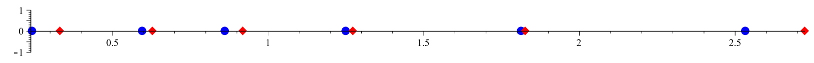

Figure: Zeros of $\mathcal{P}_6(x)$ (red diamonds) and $\mathcal{P}_6(x;-0.3)$ (blue circles)

Remark

No interlacing between co-modified polynomials and initial ones.

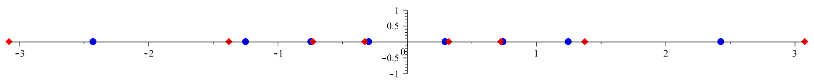

Figure: Zeros of $\mathcal{P}_9(x)$ (red diamonds) and $\mathcal{P}_8(x;0,0.6)$ (blue circles).

☞ For the co-dilated case $0 < \nu_k < 1$:
Zeros of $\mathcal{P}_n(x; \nu_k)$, $\mathcal{P}_n(x)$, $\mathcal{P}_k(x)$ and $\mathcal{P}_{k-1}(x)$ interlace in a special manner.

Example

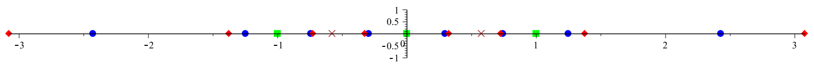

Figure: Zeros of $\mathcal{P}_9(x)$ (red diamonds), $\mathcal{P}_9(x; \nu_3 = 0.6)$ (blue circles), $\mathcal{P}_3(x)$ (green squares) and $\mathcal{P}_2(x)$ (brown cross).

☞ For the co-dilated case $0 < \nu_k < 1$:
Zeros of $\mathcal{P}_n(x;\nu_k)$, $\mathcal{P}_n(x)$, $\mathcal{P}_k(x)$ and $\mathcal{P}_{k-1}(x)$ interlace in a special manner.

Example

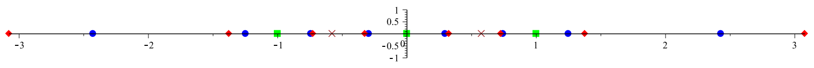

Figure: Zeros of $\mathcal{P}_9(x)$ (red diamonds), $\mathcal{P}_9(x;\nu_3 = 0.6)$ (blue circles), $\mathcal{P}_3(x)$ (green squares) and $\mathcal{P}_2(x)$ (brown cross).

Example

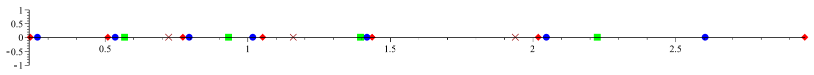

Figure: Zeros of CRR polynomials:
$\mathcal{P}_7(x)$ (red diamonds), $\mathcal{P}_7(x;\nu_4 = 0.5)$ (blue circles),
$\mathcal{P}_4(x)$ (green squares) and $\mathcal{P}_3(x)$ (brown cross).

Co-modified case

Recall

$$\mathcal{P}_n(x;\mu_k,\nu_k) = \mathcal{P}_n(x), \qquad n < k,$$
$$\mathcal{P}_n(x;\mu_k,\nu_k) = \mathcal{P}_n(x) - \mathcal{S}_k(x)\mathcal{P}_{n-k}^{(k)}(x), \qquad n \geq k,$$

- $y_{k,j}$, $j = 1, 2, \ldots, k+1$: zeros of $\mathcal{S}_k(x)$
- $x_{k,j}$, $j = 1, 2, \ldots, k$: zeros of $\mathcal{P}_k(x)$ and
- $x_{k-1,j}$, $j = 1, 2, \ldots, k-1$: zeros of $\mathcal{P}_{k-1}(x)$
- $c = \frac{\nu_{k-1}}{\mu_k}$

☞ Triple interlacing : for $c > 0$ on $\mathbb{R}\backslash[-\infty, x_{k,1}]$.

$$x_{k-1,1} < y_{k,3} < x_{k,2} < \ldots < x_{k-1,k-1} < y_{k,k+1} < x_{k,k}.$$

Example

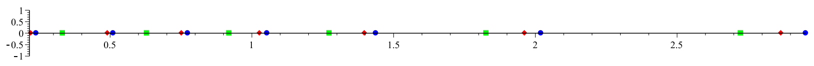

Figure: Zeros of $\mathcal{S}_7(x)$ (red diamonds) when $\mu_7 = 0.4$ and $\nu_7 = 1.2$, the CRR polynomials $\mathcal{P}_7(x)$ (blue circles) and $\mathcal{P}_6(x)$ (green squares).

☞ when $c < 0$, then on $\mathbb{R}\backslash[x_{k,k}, \infty]$,

$$x_{k,1} < y_{k,1} < x_{k-1,1} < \ldots < x_{k,k-1} < y_{k,k-1} < x_{k-1,k-1}.$$

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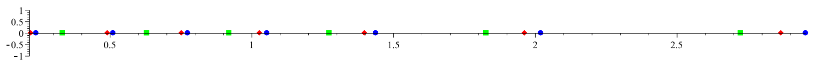
Example



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☞ when $c < 0$, then on $\mathbb{R}\backslash[x_{k,k}, \infty]$,

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Further problems

- What about other cases?

Recall

For $x \in \mathbb{R}\backslash X$, the following formulas hold:

$$\mathcal{P}_n(x;\mu_k,\nu_k) = \mathcal{P}_n(x), \qquad n < k,$$
$$\mathcal{P}_n(x;\mu_k,\nu_k) = \mathcal{P}_n(x) - \mathcal{S}_k(x)\mathcal{P}^{(k)}_{n-k}(x), \qquad n \geq k,$$

where $\mathcal{S}_k(x) = \mu_k\mathcal{P}_k(x) + (\nu_k - 1)\lambda_k(x^2+1)\mathcal{P}_{k-1}(x)$.

Recall

For $x \in \mathbb{R} \backslash X$, the following formulas hold:

$$\mathcal{P}_n(x;\mu_k,\nu_k) = \mathcal{P}_n(x), \qquad n < k,$$
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Challenges

The structural model

- ☞ has issues for finite composition of perturbations.
- ☞ does not hold in the whole of $\mathbb{R}$.

Table of Contents

Transfer matrix approach

Theorem (V. Shukla and AS)

The following relation hold in $\mathbb{R}$: (Single perturbation)

$$\prod_{j=1}^{k}\lambda_j(x^2+1)^k\begin{bmatrix}\mathcal{P}_{n+1}(x;\mu_k,\nu_k)\\-\mathcal{Q}_{n+1}(x;\mu_k,\nu_k)\end{bmatrix}=\mathbf{N}_k\begin{bmatrix}\mathcal{P}_{n+1}\\-\mathcal{Q}_{n+1}\end{bmatrix},$$

$$\mathbf{N}_k=\begin{bmatrix}\displaystyle\prod_{j=1}^{k}\lambda_j(x^2+1)^k+\mathcal{S}_k\mathcal{Q}_k & \mathcal{S}_k\mathcal{P}_k\\ \mathcal{Q}_k\hat{\mathcal{S}}_k & \displaystyle\hat{\mathcal{S}}_k\mathcal{P}_k+\prod_{j=1}^{k}\lambda_j(x^2+1)^k\end{bmatrix},$$

$$\hat{\mathcal{S}}_k(x)=-\mu_k\mathcal{Q}_k(x)-(\nu_k-1)\lambda_k(x^2+1)\mathcal{Q}_{k-1}(x),$$
$$\mathcal{S}_k(x)=\mu_k\mathcal{P}_k(x)+(\nu_k-1)\lambda_k(x^2+1)\mathcal{P}_{k-1}(x).$$

Theorem (V. Shukla and AS)

Let k, m be two fixed non-negative integer numbers with $m < k$. Then for $n > m$, the following relation holds: (Finite perturbations)

$$\prod_{j=m}^{k}\prod_{l=0}^{j}\lambda_l(x^2+1)^l\begin{bmatrix}\mathcal{P}_{n+1}(x;;\mu_m,\nu_m,\ldots\mu_k,\nu_k)\\ -\mathcal{Q}_{n+1}(x;;\mu_m,\nu_m,\ldots\mu_k,\nu_k)\end{bmatrix}=\prod_{j=m}^{k}\mathbf{N}_j\begin{bmatrix}\mathcal{P}_{n+1}\\ -\mathcal{Q}_{n+1}\end{bmatrix}.$$

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- Continued fraction

$$\mathcal{R}_{II}(x)=\frac{1}{w_0(x)}-\frac{\chi_1^L(x)\chi_1^R(x)}{w_1(x)}-\frac{\chi_2^L(x)\chi_2^R(x)}{w_2(x)}-\cdots,$$

- $(k+1)$-th approximant

$$\mathcal{R}_{II}^{k+1}(x)=\frac{1}{w_{k+1}(x)}-\frac{\chi_{k+2}^L(x)\chi_{k+2}^R(x)}{w_{k+2}(x)}-\frac{\chi_{k+3}^L(x)\chi_{k+3}^R(x)}{w_{k+3}(x)}-\cdots,$$

$$\mathcal{R}_{II}(x;\mu_k,\nu_k) = \frac{1}{w_0(x)} - \cdots - \frac{\nu_k\chi_k^L(x)\chi_k^R(x)}{w_k(x)-\mu_k} - \frac{\chi_{k+1}^L(x)\chi_{k+1}^R(x)}{w_{k+1}(x)} - \cdots$$
$$= \frac{1}{w_0(x)} - \cdots - \frac{\nu_k\chi_k^L(x)\chi_k^R(x)}{w_k(x)-\mu_k-\chi_{k+1}^L(x)\chi_{k+1}^R(x))\mathcal{R}_{II}^{k+1}(x)}.$$

[15] K. Castillo, F. Marcellán and J. Rivero, On co-polynomials on the real line and the unit circle, Operations research, engineering, and cyber security, Springer Optim. Appl., Springer, Cham, **113** (2017), 69–94.

$$\mathcal{R}_{II}(x;\mu_k,\nu_k) = \frac{1}{w_0(x)} - \cdots - \frac{\nu_k\chi_k^L(x)\chi_k^R(x)}{w_k(x)-\mu_k} - \frac{\chi_{k+1}^L(x)\chi_{k+1}^R(x)}{w_{k+1}(x)} - \cdots$$

$$= \frac{1}{w_0(x)} - \cdots - \frac{\nu_k\chi_k^L(x)\chi_k^R(x)}{w_k(x)-\mu_k-\chi_{k+1}^L(x)\chi_{k+1}^R(x))\mathcal{R}_{II}^{k+1}(x)}.$$

- Homography mapping [15]

$$r(x) = \frac{a(x)u(x)+b(x)}{c(x)u(x)+d(x)}, \quad a(x)d(x)-b(x)c(x) \neq 0,$$

- Rational spectral transformation

$$r(z) \doteq \mathbf{A}(z)u(z), \quad \text{where} \quad \mathbf{A}(z) = \begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix}.$$

[15] K. Castillo, F. Marcellán and J. Rivero, On co-polynomials on the real line and the unit circle, Operations research, engineering, and cyber security, Springer Optim. Appl., Springer, Cham, **113** (2017), 69–94.

Spectral Transformation

Theorem (V. Shukla and AS)

A relation between continued fraction for co-modified perturbed polynomial and its $(k+1)$-th approximant in homography mapping representation can be given in the following way:

$$\mathcal{R}_{II}(x;\mu_k,\nu_k) = \frac{\mathcal{A}(x)\mathcal{R}_{II}^{k+1}(x) + \mathcal{B}(x)}{\mathcal{C}(x)\mathcal{R}_{II}^{k+1}(x) + \mathcal{D}(x)},$$

where

$$\mathcal{A}(x) = \chi_{k+1}^{L}(x)\chi_{k+1}^{R}(x)\mathcal{Q}_k,$$

$$\mathcal{B}(x) = -\mathcal{Q}_{k+1} + \mu_k\mathcal{Q}_k + (\nu_k - 1)\chi_k^{L}(x)\chi_k^{R}(x)\mathcal{Q}_{k-1},$$

$$\mathcal{C}(x) = \chi_{k+1}^{L}(x)\chi_{k+1}^{R}(x)\mathcal{P}_k,$$

$$D(x) = -\mathcal{P}_{k+1} + \mu_k\mathcal{P}_k + (\nu_k - 1)\chi_k^{L}(x)\chi_k^{R}(x)\mathcal{P}_{k-1}.$$

Corollary

The new function $\mathcal{R}_{II}(x;\mu_k,\nu_k)$ *can be seen as a spectral transformation of* $\mathcal{R}_{II}^{k+1}(x)$*, the* $(k+1)$*-th approximant, precisely, we have*

$$\mathcal{R}_{II}(x;\mu_k,\nu_k) \doteq \begin{bmatrix} \chi_{k+1}^{L}(x)\chi_{k+1}^{R}(x)\mathcal{Q}_k & -\mathcal{Q}_{k+1}-\hat{\mathcal{S}}_k \\ \chi_{k+1}^{L}(x)\chi_{k+1}^{R}(x)\mathcal{P}_k & -\mathcal{P}_{k+1}+\mathcal{S}_k \end{bmatrix} \mathcal{R}_{II}^{k+1}(x)$$

Corollary

The new function $\mathcal{R}_{II}(x;\mu_k,\nu_k)$ can be seen as a spectral transformation of $\mathcal{R}_{II}^{k+1}(x)$, the $(k+1)$-th approximant, precisely, we have

$$\mathcal{R}_{II}(x;\mu_k,\nu_k) \doteq \begin{bmatrix} \chi_{k+1}^{L}(x)\chi_{k+1}^{R}(x)\mathcal{Q}_k & -\mathcal{Q}_{k+1}-\hat{\mathcal{S}}_k \\ \chi_{k+1}^{L}(x)\chi_{k+1}^{R}(x)\mathcal{P}_k & -\mathcal{P}_{k+1}+\mathcal{S}_k \end{bmatrix} \mathcal{R}_{II}^{k+1}(x)$$

Corollary

If we take $\mu_k = 0$ and $\nu_k = 1$, then $\mathcal{R}_{II}(x) = \mathcal{R}_{II}(x;\mu_k,\nu_k)$, and above relation gives

$$\chi_{k+1}^{L}(x)\chi_{k+1}^{R}(x)\mathcal{R}_{II}^{k+1}(x) = \frac{\mathcal{P}_{k+1}\mathcal{R}_{II}(x)-\mathcal{Q}_{k+1}}{\mathcal{P}_k\mathcal{R}_{II}(x)-\mathcal{Q}_k}.$$

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Corollary

If we take $\mu_k = 0$ *and* $\nu_k = 1$*, then* $\mathcal{R}_{II}(x) = \mathcal{R}_{II}(x;\mu_k,\nu_k)$*, and above relation gives*

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A question

- ✷ What about the spectral transformation between Co-modified and the Original one?

Theorem (V. Shukla and AS)

Following expression gives a relation between $\mathcal{R}_{II}(x;\mu_k,\nu_k)$ and $\mathcal{R}_{II}(x)$.

$$\mathcal{R}_{II}(x;\mu_k,\nu_k) = \frac{\mathcal{A}_{\mu,\nu}(x)\mathcal{R}_{II}(x) + \mathcal{B}_{\mu,\nu}(x)}{\mathcal{C}_{\mu,\nu}(x)\mathcal{R}_{II}(x) + \mathcal{D}_{\mu,\nu}(x)},$$

$$\mathcal{A}_{\mu,\nu}(x) = \prod_{j=1}^{k} \chi_j^L(x)\chi_j^R(x) - (\nu_k - 1)\chi_k^L(x)\chi_k^R(x)\mathcal{Q}_{k-1}\mathcal{P}_k - \mu_k\mathcal{Q}_k\mathcal{P}_k,$$

$$\mathcal{B}_{\mu,\nu}(x) = (\nu_k - 1)\chi_k^L(x)\chi_k^R(x)\mathcal{Q}_{k-1}\mathcal{Q}_k - \mu_k[\mathcal{Q}_k]^2,$$

$$\mathcal{D}_{\mu,\nu} = \prod_{j=1}^{k} \chi_j^L(x)\chi_j^R(x) + (\nu_k - 1)\chi_k^L(x)\chi_k^R(x)\mathcal{P}_{k-1}\mathcal{Q}_k + \mu_k\mathcal{Q}_k\mathcal{P}_k,$$

$$\mathcal{C}_{\mu,\nu}(x) = -(\nu_k - 1)\chi_k^L(x)\chi_k^R(x)\mathcal{P}_{k-1}\mathcal{P}_k - \mu_k[\mathcal{P}_k]^2.$$

Corollary

$\mathcal{R}_{II}(x;\mu_k,\nu_k)$ defines a rational spectral transformation of $\mathcal{R}_{II}(x)$ as

$$\mathcal{R}_{II}(x;\mu_k,\nu_k) \doteq \begin{bmatrix} \prod_{j=1}^{k}\chi_j^L\chi_j^R + \hat{\mathcal{S}}_k\mathcal{P}_k & -\hat{\mathcal{S}}_k\mathcal{Q}_k \\ -\mathcal{S}_k\mathcal{P}_k & \prod_{j=1}^{k}\chi_j^L\chi_j^R + \mathcal{S}_k\mathcal{P}_k \end{bmatrix}\mathcal{R}_{II}(x).$$

Corollary

$\mathcal{R}_{II}(x;\mu_k,\nu_k)$ *defines a rational spectral transformation of* $\mathcal{R}_{II}(x)$ *as*

$$\mathcal{R}_{II}(x;\mu_k,\nu_k)\doteq\begin{bmatrix}\prod_{j=1}^{k}\chi_j^L\chi_j^R+\hat{\mathcal{S}}_k\mathcal{P}_k & -\hat{\mathcal{S}}_k\mathcal{Q}_k\\ -\mathcal{S}_k\mathcal{P}_k & \prod_{j=1}^{k}\chi_j^L\chi_j^R+\mathcal{S}_k\mathcal{P}_k\end{bmatrix}\mathcal{R}_{II}(x).$$

Theorem (V. Shukla and AS)

$\mathcal{R}_{II}(x;\mu_k,\nu_k)$ *is a pure rational spectral transformation of* $\mathcal{R}_{II}(x)$ *given by*

$$\mathcal{R}_{II}(x;\mu_k,\nu_k)=cof(\mathbf{N}_k)\mathcal{R}_{II}(x),$$

where cof(.) is the cofactor matrix operator.

Further problems

- What about the biorthogonality [16] relation (in the sense of Konhauser) for the perturbed one?
- The sequences $\{\mathcal{U}_m(x)\}_{m=0}^{\infty}$ and $\{\mathcal{V}_n(x)\}_{n=0}^{\infty}$ are biorthogonal[17] over the interval (a, b) with respect to the weight function $\omega(x)$ provided

$$\int_a^b \mathcal{U}_m(x)\mathcal{V}_n(x)\omega(x)dx \begin{cases} = 0, & m, n = 0, 1, \cdots, m \neq n; \\ \neq 0, & m = n, \end{cases}$$

are satisfied.

[16] K. K. Behera and A. Swaminathan, Biorthogonal rational functions of R_{II} type, Proc. Amer. Math. Soc., **147** (7), 2019 3061-3073

[17] J. D. E. Konhauser, Some properties of biothogonal polynomials, J. Math. Anal. Appl. **11** (1965), 242–260.

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Thank You