Determinantal Formulas for Exceptional Orthogonal Polynomials

Brian Simanek

Baylor Analysis Fest
Waco, TX

May 27, 2022

Supported by Simons Foundation Grant 707882

XOP

- eXceptional Orthogonal Polynomials

XOP

- eXceptional Orthogonal Polynomials
- Polynomial eigenfunctions of second order linear differential operators with rational coefficients.

XOP

- eXceptional Orthogonal Polynomials
- Polynomial eigenfunctions of second order linear differential operators with rational coefficients.
- They exist outside the classical Bochner classification because they omit polynomials of certain degrees.

Darboux Transformations

- Start with a second order linear differential operator T with rational coefficients.

Darboux Transformations

- Start with a second order linear differential operator T with rational coefficients.
- Write it as

$$T = BA + \lambda$$

where A, B are first order linear differential operators with rational coefficients and $\lambda \in \mathbb{C}$.

Darboux Transformations

- Start with a second order linear differential operator T with rational coefficients.
- Write it as
$$T = BA + \lambda$$
where A, B are first order linear differential operators with rational coefficients and $\lambda \in \mathbb{C}$.
- The operator
$$AB + \lambda$$
is a rational Darboux transformation of T.

Darboux Transformations

- Start with a second order linear differential operator T with rational coefficients.
- Write it as
$$T = BA + \lambda$$
where A, B are first order linear differential operators with rational coefficients and $\lambda \in \mathbb{C}$.
- The operator
$$AB + \lambda$$
is a rational Darboux transformation of T.
- The factorization of T in this form is not unique.

Classification Theorem

Theorem (García-Ferrero, Gómez-Ullate, & Milson (2019), JMAA)

Every exceptional orthogonal polynomial system can be obtained by applying a finite sequence of Darboux transformations to a classical orthogonal polynomial system.

Classification Theorem

Theorem (García-Ferrero, Gómez-Ullate, & Milson (2019), JMAA)

Every exceptional orthogonal polynomial system can be obtained by applying a finite sequence of Darboux transformations to a classical orthogonal polynomial system.

- All exceptional orthogonal polynomial sequences are labeled as Exceptional Jacobi, Exceptional Laguerre, or Exceptional Hermite.

Properties

- What properties are common to classical orthogonal polynomials and XOP and what is different?

Properties

- What properties are common to classical orthogonal polynomials and XOP and what is different?
- Recurrence relations - they exist but in a different form (Duran)

Properties

- What properties are common to classical orthogonal polynomials and XOP and what is different?
- Recurrence relations - they exist but in a different form (Duran)
- Zeros... its complicated.

Properties

- What properties are common to classical orthogonal polynomials and XOP and what is different?
- Recurrence relations - they exist but in a different form (Duran)
- Zeros... its complicated.
- Determinantal formulas
  - previous XOP work by Duran, Kelly, Liaw, Osborn

OPRL Determinantal Formulas

$$M_n = \begin{pmatrix} & & \displaystyle\int t^{i+j-2}\, d\mu(t) & & \\ 1 & x & \cdots & \cdots & x^n \end{pmatrix}$$

OPRL Determinantal Formulas

$$M_n = \begin{pmatrix} & & \int t^{i+j-2} d\mu(t) & & \\ & & & & \\ 1 & x & \cdots & \cdots & x^n \end{pmatrix}$$

$$\det(M_n) = P_n(x;\mu)$$

OPRL Determinantal Formulas

$$M_n = \begin{pmatrix} & \int t^{i+j-2} d\mu(t) & & \\ & & & \\ 1 \quad x & \cdots & \cdots & x^n \end{pmatrix}$$

$$\det(M_n) = P_n(x;\mu)$$

- Analogous XOP determinantal formulas for X_m polynomials when m is small by Kelly, Liaw, and Osborn.

OPRL Determinantal Formulas

$$T_n = \begin{pmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^n \\ 1 & z_2 & z_2^2 & \cdots & z_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_n & z_n^2 & \cdots & z_n^n \\ 1 & x & \cdots & \cdots & x^n \end{pmatrix}$$

OPRL Determinantal Formulas

$$T_n = \begin{pmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^n \\ 1 & z_2 & z_2^2 & \cdots & z_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_n & z_n^2 & \cdots & z_n^n \\ 1 & x & \cdots & \cdots & x^n \end{pmatrix}$$

$$\det(T_n) = C \prod_{j=1}^{n} (x - z_j)$$

and C is a Vandermonde determinant

OPRL Formulas

$$P_n^{(\alpha,\beta)}(x) = \sum_{j=0}^{n} \binom{n+\alpha}{n-j} \binom{n+\beta}{j} \left(\frac{x-1}{2}\right)^j \left(\frac{x+1}{2}\right)^{n-j}$$

$$L_n^{\alpha}(x) = \sum_{j=0}^{n} \binom{n+\alpha}{n-j} \frac{(-x)^j}{j!}$$

X_m-Jacobi Polynomials

- Depend on $m \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}$ and require
  - $\beta > 0$
  - $\alpha + 1 - m > 0$
  - $\alpha + 1 - m - \beta \notin \{0, 1, \ldots, m-1\}$

X_m-Jacobi Polynomials

- Depend on $m \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}$ and require
  - $\beta > 0$
  - $\alpha + 1 - m > 0$
  - $\alpha + 1 - m - \beta \notin \{0, 1, \ldots, m-1\}$
- If $n \geq m$, then $P_{m,n}^{(\alpha,\beta)}(x)$ is a constant multiple of

$$\frac{(x-1)(1+\alpha+\beta+n-m)}{2} P_m^{(-\alpha-1,\beta-1)}(x) P_{n-m-1}^{(\alpha+2,\beta)}(x) + (\alpha+1-m) P_m^{(-\alpha-2,\beta)}(x) P_{n-m}^{(\alpha+1,\beta-1)}(x)$$

X_m-Jacobi Polynomials

- Depend on $m \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}$ and require
  - $\beta > 0$
  - $\alpha + 1 - m > 0$
  - $\alpha + 1 - m - \beta \notin \{0, 1, \ldots, m-1\}$
- If $n \geq m$, then $P_{m,n}^{(\alpha,\beta)}(x)$ is a constant multiple of

$$\frac{(x-1)(1+\alpha+\beta+n-m)}{2} P_m^{(-\alpha-1,\beta-1)}(x) P_{n-m-1}^{(\alpha+2,\beta)}(x) + (\alpha+1-m) P_m^{(-\alpha-2,\beta)}(x) P_{n-m}^{(\alpha+1,\beta-1)}(x)$$

- Most zeros in $(-1, 1)$, but m can be outside this interval.

New Formula

- Let $\{z_{j,N}^{(\alpha,\beta)}\}_{j=1}^{N}$ denote the zeros of $P_N^{(\alpha,\beta)}$ and

$$\{W_{j,m,n}^{(\alpha,\beta)}\}_{j=1}^{n} = \{z_{j,m}^{(-\alpha-1,\beta-1)}\}_{j=1}^{m} \bigcup \{z_{k,n-m}^{(\alpha+1,\beta-1)}\}_{k=1}^{n-m},$$

which consists of n distinct points.

New Formula

- Let $\{z_{j,N}^{(\alpha,\beta)}\}_{j=1}^{N}$ denote the zeros of $P_N^{(\alpha,\beta)}$ and

$$\{W_{j,m,n}^{(\alpha,\beta)}\}_{j=1}^{n} = \{z_{j,m}^{(-\alpha-1,\beta-1)}\}_{j=1}^{m} \bigcup \{z_{k,n-m}^{(\alpha+1,\beta-1)}\}_{k=1}^{n-m},$$

which consists of n distinct points.

- $$R_n(x,\beta) = (-1)^n P_n^{(-n-1,\beta)}(1+2x).$$

New Formula

- Let $\{z_{j,N}^{(\alpha,\beta)}\}_{j=1}^{N}$ denote the zeros of $P_N^{(\alpha,\beta)}$ and
$$\{W_{j,m,n}^{(\alpha,\beta)}\}_{j=1}^{n} = \{z_{j,m}^{(-\alpha-1,\beta-1)}\}_{j=1}^{m} \bigcup \{z_{k,n-m}^{(\alpha+1,\beta-1)}\}_{k=1}^{n-m},$$
which consists of n distinct points.

- $$R_n(x,\beta) = (-1)^n P_n^{(-n-1,\beta)}(1+2x).$$

- Let M_n be the $(n+1)\times(n+1)$ matrix whose j^{th} row is
$$\left[1, W_{j,m,n}^{(\alpha,\beta)}, \left(W_{j,m,n}^{(\alpha,\beta)}\right)^2, \ldots, \left(W_{j,m,n}^{(\alpha,\beta)}\right)^n\right]$$
for $j = 1, \ldots, n$ and let the last row of M_n be
$$\left[\frac{1}{\beta^{(1)}}, \frac{-1}{\beta^{(2)}} R_1(x), \frac{2!}{\beta^{(3)}} R_2(x), \ldots, \frac{(-1)^n n!}{\beta^{(n+1)}} R_n(x)\right].$$

New Formula

Theorem (S. 2022, submitted)

For $n \geq m$,

$$P_{m,n}^{(\alpha,\beta)}(x) = \frac{C_{n,m,\alpha,\beta}}{\prod_{i<j}(W_{j,m,n}^{(\alpha,\beta)} - W_{i,m,n}^{(\alpha,\beta)})} \det(M_n)$$

Exceptional Laguerre Polynomials

- Type I: $\alpha > 0$, $m \in \mathbb{N}$, $n \geq m$

$$L_{m,n}^{I,\alpha}(x) = L_m^{\alpha}(-x)L_{n-m}^{\alpha-1}(x) + L_m^{\alpha-1}(-x)L_{n-m-1}^{\alpha}(x)$$

Exceptional Laguerre Polynomials

- Type I: $\alpha > 0$, $m \in \mathbb{N}$, $n \geq m$

$$L^{I,\alpha}_{m,n}(x) = L^{\alpha}_{m}(-x)L^{\alpha-1}_{n-m}(x) + L^{\alpha-1}_{m}(-x)L^{\alpha}_{n-m-1}(x)$$

- Type II: $\alpha > m-1$, $m \in \mathbb{N}$, $n \geq m$

$$L^{II,\alpha}_{m,n}(x) = xL^{-\alpha-1}_{m}(x)L^{\alpha+2}_{n-m-1}(x)+(m-\alpha-1)L^{-\alpha-2}_{m}(x)L^{\alpha+1}_{n-m}(x)$$

Exceptional Laguerre Polynomials

- Type I: $\alpha > 0,\ m \in \mathbb{N},\ n \geq m$

$$L^{I,\alpha}_{m,n}(x) = L^{\alpha}_{m}(-x)L^{\alpha-1}_{n-m}(x) + L^{\alpha-1}_{m}(-x)L^{\alpha}_{n-m-1}(x)$$

- Type II: $\alpha > m-1,\ m \in \mathbb{N},\ n \geq m$

$$L^{II,\alpha}_{m,n}(x) = xL^{-\alpha-1}_{m}(x)L^{\alpha+2}_{n-m-1}(x)+(m-\alpha-1)L^{-\alpha-2}_{m}(x)L^{\alpha+1}_{n-m}(x)$$

- Type III: $\alpha \in (-1,0),\ m \in \mathbb{N},\ n = 0$ or $n > m$

$$L^{III,\alpha}_{m,n}(x) = xL^{\alpha+2}_{n-m-2}(x)L^{-\alpha-1}_{m}(-x)+(m+1)L^{-\alpha-2}_{m+1}(-x)L^{\alpha+1}_{n-m-1}(x)$$

Exceptional Laguerre Polynomials

- Type I: $\alpha > 0$, $m \in \mathbb{N}$, $n \geq m$

$$L^{I,\alpha}_{m,n}(x) = L^{\alpha}_{m}(-x)L^{\alpha-1}_{n-m}(x) + L^{\alpha-1}_{m}(-x)L^{\alpha}_{n-m-1}(x)$$

- Type II: $\alpha > m-1$, $m \in \mathbb{N}$, $n \geq m$

$$L^{II,\alpha}_{m,n}(x) = xL^{-\alpha-1}_{m}(x)L^{\alpha+2}_{n-m-1}(x)+(m-\alpha-1)L^{-\alpha-2}_{m}(x)L^{\alpha+1}_{n-m}(x)$$

- Type III: $\alpha \in (-1,0)$, $m \in \mathbb{N}$, $n = 0$ or $n > m$

$$L^{III,\alpha}_{m,n}(x) = xL^{\alpha+2}_{n-m-2}(x)L^{-\alpha-1}_{m}(-x)+(m+1)L^{-\alpha-2}_{m+1}(-x)L^{\alpha+1}_{n-m-1}(x)$$

- For any $\tau \in \mathbb{C}$, let $\{x^{(\tau)}_{j,N}\}^{N}_{j=1}$ denote the zeros of L^{τ}_{N}.

New Formula (Type I)

Theorem (S. 2022, submitted)

$$\{X_{j,m,n}^{(\alpha)}\}_{j=1}^{n} = \{-x_{j,m}^{(\alpha-1)}\}_{j=1}^{m} \bigcup \{x_{k,n-m}^{(\alpha-1)}\}_{k=1}^{n-m}.$$

New Formula (Type I)

Theorem (S. 2022, submitted)

$$\{X_{j,m,n}^{(\alpha)}\}_{j=1}^{n} = \{-x_{j,m}^{(\alpha-1)}\}_{j=1}^{m} \bigcup \{x_{k,n-m}^{(\alpha-1)}\}_{k=1}^{n-m}.$$

Let M_n^I be the $(n+1)\times(n+1)$ matrix whose j^{th} row is

$$\left[1, \left(X_{j,m,n}^{(\alpha)}\right), \left(X_{j,m,n}^{(\alpha)}\right)^2, \ldots, \left(X_{j,m,n}^{(\alpha)}\right)^n\right]$$

for $j = 1, \ldots, n$ and let the last row of M_n^I be

$$\left[\frac{1}{\alpha}, \frac{x}{1+\alpha}, \frac{x^2}{2+\alpha}, \ldots, \frac{x^n}{n+\alpha}\right].$$

New Formula (Type I)

Theorem (S. 2022, submitted)

$$\{X_{j,m,n}^{(\alpha)}\}_{j=1}^{n} = \{-x_{j,m}^{(\alpha-1)}\}_{j=1}^{m} \bigcup \{x_{k,n-m}^{(\alpha-1)}\}_{k=1}^{n-m}.$$

Let M_n^I be the $(n+1)\times(n+1)$ matrix whose j^{th} row is

$$\left[1, \left(X_{j,m,n}^{(\alpha)}\right), \left(X_{j,m,n}^{(\alpha)}\right)^2, \dots, \left(X_{j,m,n}^{(\alpha)}\right)^n\right]$$

for $j = 1, \dots, n$ and let the last row of M_n^I be

$$\left[\frac{1}{\alpha}, \frac{x}{1+\alpha}, \frac{x^2}{2+\alpha}, \dots, \frac{x^n}{n+\alpha}\right].$$

$$L_{m,n}^{I,\alpha}(x) = \frac{C_{n,m,\alpha}}{\prod_{1\le i<j\le n}(X_{j,m,n}^{(\alpha)} - X_{i,m,n}^{(\alpha)})} \det(M_n^I)$$

New Formula (Type II)

Theorem (S. 2022, submitted)

$$\{Y_{j,m,n}^{(\alpha)}\}_{j=1}^{n} = \{x_{j,m}^{(-\alpha-1)}\}_{j=1}^{m} \bigcup \{x_{k,n-m}^{(\alpha+1)}\}_{k=1}^{n-m}.$$

Let M_n^{II} *be the* $(n+1) \times (n+1)$ *matrix whose* j^{th} *row is*

$$\left[1, Y_{j,m,n}^{(\alpha)}, \left(Y_{j,m,n}^{(\alpha)}\right)^2, \ldots, \left(Y_{j,m,n}^{(\alpha)}\right)^n\right]$$

for $j = 1, \ldots, n$ *and let the last row of* M_n^{II} *be*

$$[E_0(x), E_1(x), 2!E_2(x), 3!E_3(x) \ldots, n!E_n(x)]$$

where $E_n(x) = (-1)^n L_n^{-n-1}(x)$.

New Formula (Type II)

Theorem (S. 2022, submitted)

$$\{Y_{j,m,n}^{(\alpha)}\}_{j=1}^{n} = \{x_{j,m}^{(-\alpha-1)}\}_{j=1}^{m} \bigcup \{x_{k,n-m}^{(\alpha+1)}\}_{k=1}^{n-m}.$$

Let M_n^{II} be the $(n+1)\times(n+1)$ matrix whose j^{th} row is

$$\left[1, Y_{j,m,n}^{(\alpha)}, \left(Y_{j,m,n}^{(\alpha)}\right)^2, \ldots, \left(Y_{j,m,n}^{(\alpha)}\right)^n\right]$$

for $j = 1, \ldots, n$ and let the last row of M_n^{II} be

$$[E_0(x), E_1(x), 2!E_2(x), 3!E_3(x) \ldots, n!E_n(x)]$$

where $E_n(x) = (-1)^n L_n^{-n-1}(x)$.

$$L_{m,n}^{II,\alpha}(x) = \frac{C_{n,m,\alpha}}{\prod_{1\le i<j\le n}(Y_{j,m,n}^{(\alpha)} - Y_{i,m,n}^{(\alpha)})} \det(M_n^{II})$$

New Formula (Type III)

Theorem (S. 2022, submitted)

$$\{Z^{(\alpha)}_{j,m,n}\}_{j=1}^{n-1} = \{-x^{(-\alpha-1)}_{j,m}\}_{j=1}^{m} \bigcup \{x^{(\alpha+1)}_{k,n-m-1}\}_{k=1}^{n-m-1}.$$

Let M_n^{III} be the $n \times n$ matrix whose j^{th} row is

$$\left[1, \left(Z^{(\alpha)}_{j,m,n}\right), \left(Z^{(\alpha)}_{j,m,n}\right)^2, \ldots, \left(Z^{(\alpha)}_{j,m,n}\right)^{n-1}\right]$$

for $j = 1, \ldots, n-1$ and let the last row of M_n^{III} be

$$\left[1, \frac{x}{2}, \ldots, \frac{x^{n-1}}{n}\right].$$

$$L^{III,\alpha}_{m,n}(x) = \frac{xC_{n,m,\alpha}}{\prod_{1\le i<j\le n-1}(Z^{(\alpha)}_{j,m,n} - Z^{(\alpha)}_{i,m,n})}\det(M_n^{III}) + L^{III,\alpha}_{m,n}(0)$$

Exceptional Hermite Polynomials

- Define a partition $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{N}^m$ where

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1$$

$|\lambda| = \lambda_1 + \cdots + \lambda_m.$

Exceptional Hermite Polynomials

- Define a partition $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{N}^m$ where

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1$$

  $|\lambda| = \lambda_1 + \cdots + \lambda_m$.

- The generalized Hermite polynomial H_λ is

$$H_\lambda = \mathrm{Wr}[H_{\lambda_m}, H_{\lambda_{m-1}+1}, \dots, H_{\lambda_2+m-2}, H_{\lambda_1+m-1}],$$

Exceptional Hermite Polynomials

- Define a partition $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{N}^m$ where

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1$$

$|\lambda| = \lambda_1 + \cdots + \lambda_m$.

- The generalized Hermite polynomial H_λ is

$$H_\lambda = \mathrm{Wr}[H_{\lambda_m}, H_{\lambda_{m-1}+1}, \ldots, H_{\lambda_2+m-2}, H_{\lambda_1+m-1}],$$

- Exceptional Hermite polynomial $H^{(\lambda)}_{|\lambda|,n}$

$$H^{(\lambda)}_{|\lambda|,n} = \mathrm{Wr}[H_{\lambda_m}, H_{\lambda_{m-1}+1}, \ldots, H_{\lambda_2+m-2}, H_{\lambda_1+m-1}, H_{n-|\lambda|+m}]$$

New Formula

Theorem

Let $\{y_{j,N}\}_{j=1}^{N}$ *denote the zeros of* H_N *and let*

$$\{U_{j,n}\}_{j=1}^{n-1} = \left\{\frac{i}{\sqrt{2}}, \frac{-i}{\sqrt{2}}\right\} \bigcup \{y_{k,n-3}\}_{k=1}^{n-3}.$$

Let $M_n^{\{1,1\}}$ *be the* $n \times n$ *matrix whose* j^{th} *row is*

$$\left[1, (U_{j,n}), (U_{j,n})^2, \ldots, (U_{j,n})^{n-1}\right]$$

for $j = 1, \ldots, n-1$ *and let the last row of* $M_n^{\{1,1\}}$ *be*

$$\left[1, x/2, \ldots, x^{n-1}/n\right]$$

New Formula

Theorem

Let $\{y_{j,N}\}_{j=1}^{N}$ *denote the zeros of* H_N *and let*

$$\{U_{j,n}\}_{j=1}^{n-1} = \left\{ \frac{i}{\sqrt{2}}, \frac{-i}{\sqrt{2}} \right\} \bigcup \{y_{k,n-3}\}_{k=1}^{n-3}.$$

Let $M_n^{\{1,1\}}$ *be the* $n \times n$ *matrix whose* j^{th} *row is*

$$\left[1, (U_{j,n}), (U_{j,n})^2, \ldots, (U_{j,n})^{n-1}\right]$$

for $j = 1, \ldots, n-1$ *and let the last row of* $M_n^{\{1,1\}}$ *be*

$$\left[1, x/2, \ldots, x^{n-1}/n\right]$$

$$H_{2,n}^{(\{1,1\})}(x) = x \cdot C_n \prod_{1 \le i < j \le n-1} (U_{j,n} - U_{i,n})^{-1} \det(M_n^{\{1,1\}}) + H_{2,n}^{(\{1,1\})}(0)$$

Proof for $L_{m,n}^{II,\alpha}$

- Recall

$$\{Y_{j,m,n}^{(\alpha)}\}_{j=1}^{n} = \{x_{j,m}^{(-\alpha-1)}\}_{j=1}^{m} \bigcup \{x_{k,n-m}^{(\alpha+1)}\}_{k=1}^{n-m}$$

Proof for $L^{II,\alpha}_{m,n}$

- Recall

$$\{Y^{(\alpha)}_{j,m,n}\}_{j=1}^{n} = \{x^{(-\alpha-1)}_{j,m}\}_{j=1}^{m} \bigcup \{x^{(\alpha+1)}_{k,n-m}\}_{k=1}^{n-m}$$

- Let $\tilde{M}^{II}_{n}$ be the $(n+1)\times(n+1)$ matrix whose j^{th} row is

$$\left[1, (Y^{(\alpha)}_{j,m,n}-1), \left(Y^{(\alpha)}_{j,m,n}\right)(Y^{(\alpha)}_{j,m,n}-2), \ldots, \left(Y^{(\alpha)}_{j,m,n}\right)^{n-1}(Y^{(\alpha)}_{j,m,n}-n)\right]$$

for $j = 1, \ldots, n$ and with last row $[1, x, x^2, \ldots, x^n]$.

Proof for $L^{II,\alpha}_{m,n}$

- Recall

$$\{Y^{(\alpha)}_{j,m,n}\}_{j=1}^{n} = \{x^{(-\alpha-1)}_{j,m}\}_{j=1}^{m} \bigcup \{x^{(\alpha+1)}_{k,n-m}\}_{k=1}^{n-m}$$

- Let $\tilde{M}^{II}_n$ be the $(n+1) \times (n+1)$ matrix whose j^{th} row is

$$\left[1, (Y^{(\alpha)}_{j,m,n} - 1), \left(Y^{(\alpha)}_{j,m,n}\right)(Y^{(\alpha)}_{j,m,n} - 2), \ldots, \left(Y^{(\alpha)}_{j,m,n}\right)^{n-1}(Y^{(\alpha)}_{j,m,n} - n)\right]$$

for $j = 1, \ldots, n$ and with last row $[1, x, x^2, \ldots, x^n]$.

- The exceptional condition:

$$\frac{d}{dx} L^{II,\alpha}_{m,n}(x) - L^{II,\alpha}_{m,n}(x) = (\alpha + n + 1 - 2m) L^{-\alpha-1}_{m}(x) L^{\alpha+1}_{n-m}(x)$$

Proof for $L_{m,n}^{II,\alpha}$

- Write

$$L_{m,n}^{II,\alpha}(x) = \sum_{j=0}^{n} c_{j,n} x^j .$$

Proof for $L_{m,n}^{II,\alpha}$

- Write
$$L_{m,n}^{II,\alpha}(x) = \sum_{j=0}^{n} c_{j,n} x^j.$$
- If
$$\vec{c} = \begin{pmatrix} c_{0,n} \\ c_{1,n} \\ \vdots \\ c_{n,n} \end{pmatrix}, \qquad \vec{b}_{II} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ L_{m,n}^{II,\alpha}(x) \end{pmatrix},$$
then $\tilde{M}_n^{II} \vec{c} = \vec{b}_{II}$.

Proof for $L_{m,n}^{II,\alpha}$

- Write
$$L_{m,n}^{II,\alpha}(x) = \sum_{j=0}^{n} c_{j,n} x^j .$$
- If
$$\vec{c} = \begin{pmatrix} c_{0,n} \\ c_{1,n} \\ \vdots \\ c_{n,n} \end{pmatrix}, \qquad \vec{b}_{II} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ L_{m,n}^{II,\alpha}(x) \end{pmatrix},$$
then $\tilde{M}_n^{II} \vec{c} = \vec{b}_{II}$.
- $\det(\tilde{M}_n^{II})$ has the same zeros as $L_{m,n}^{II,\alpha}(x)$.

Proof for $L^{II,\alpha}_{m,n}$

- Write

$$L^{II,\alpha}_{m,n}(x) = \sum_{j=0}^{n} c_{j,n} x^j.$$

- If

$$\vec{c} = \begin{pmatrix} c_{0,n} \\ c_{1,n} \\ \vdots \\ c_{n,n} \end{pmatrix}, \qquad \vec{b}_{II} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ L^{II,\alpha}_{m,n}(x) \end{pmatrix},$$

then $\tilde{M}^{II}_n \vec{c} = \vec{b}_{II}$.

- $\det(\tilde{M}^{II}_n)$ has the same zeros as $L^{II,\alpha}_{m,n}(x)$.
- Column operations make $\tilde{M}^{II}_n$ into M^{II}_n.

Thank you for your attention.

Happy Birthday Lance!