The Birman Sequence of Integral Inequalities and Some of its Refinements

Michael Pang (University of Missouri-Columbia, MO, USA)

Based on various joint work with

Fritz Gesztesy (Baylor University, TX, USA)

Lance Littlejohn (Baylor University, TX, USA)

Isaac Michael (Louisiana State University, LA, USA)

Baylor Analysis Fest

May 23–27, 2022

Introduction

Birman presented in 1961 (almost in passing) the following sequence of inequalities (English translation in AMS Transl. (2) **53**, 23–80 (1966)):

$$\int_0^\infty \left|f^{(m)}(x)\right|^2 dx \geq \frac{[(2m-1)!!]^2}{2^{2m}} \int_0^\infty x^{-2m}|f(x)|^2\, dx$$

$$m \in \mathbb{N}, f \in C_0^\infty((0,\infty))$$

where

$$(2m-1)!! = 1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2m-1).$$

It is known that the constants $[(2m-1)!!]^2/2^{2m}$ are sharp and the inequalities are strict, i.e., equality occurs if and only if $f \equiv 0$.

More generally we have the following power weighted version of Birman's sequence:

$$\int_0^\infty x^\alpha \left| f^{(m)}(x) \right|^2 dx \geq A(m,\alpha) \int_0^\infty x^{\alpha-2m} |f(x)|^2 \, dx$$

$$m \in \mathbb{N}, f \in C_0^\infty((0,\infty))$$

where

$$A(m,\alpha) = \prod_{j=1}^{m} \left(\frac{2j-1-\alpha}{2} \right)^2 .$$

Again the constant $A(m,\alpha)$ is sharp and the inequality is strict, i.e., equality occurs if and only if $f \equiv 0$.

Notice that

$$A(m,\alpha) = 0 \qquad \text{if} \qquad \alpha \in \{2j-1 : j = 1, \cdots, m\}.$$

This is one motivation for inequalities of the form

$$\int_0^\rho x^\alpha \left|f^{(m)}(x)\right|^2 dx$$
$$\geq A(m,\alpha) \int_0^\rho x^{\alpha-2m} |f(x)|^2 \, dx + \text{positive refinement terms}$$

$$m \in \mathbb{N}, \quad f \in C_0^\infty((0,\rho)), \quad \rho > 0.$$

More about this in a few minutes.

In higher dimensions **Davies-Hinz** proved (1998):

Let $m \in \mathbb{N}$ and $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$. Then,

$$\int_{\mathbb{R}^n} d^n x\, |x|^\alpha \big|\big((-\Delta)^m f\big)(x)\big|^2$$
$$\geq A(2m, n, \alpha) \int_{\mathbb{R}^n} d^n x\, |x|^{\alpha - 4m} |f(x)|^2, \quad \alpha \in [4m - n, 2],$$

$$\int_{\mathbb{R}^n} d^n x\, |x|^\alpha \big|\big(\nabla(-\Delta)^m f\big)(x)\big|^2$$
$$\geq A(2m+1, n, \alpha) \int_{\mathbb{R}^n} d^n x\, |x|^{\alpha - 2 - 4m} |f(x)|^2, \quad \alpha \in [4m + 2 - n, 4].$$

where

$$A(2m, n, \alpha)$$
$$= \left[\prod_{j=0}^{m-1} [(n/2) + 2j - 2m + (\alpha/2)]^2 \right]$$
$$\times \left[\prod_{k=1}^{m} [(n/2) - 2k + 2m - (\alpha/2)]^2 \right],$$

and

$$A(2m+1, n, \alpha)$$
$$= \left[\prod_{j=0}^{m} [(n/2) + 2j - 2m - 1 + (\alpha/2)]^2 \right]$$
$$\times \left[\prod_{k=1}^{m} [(n/2) - 2k + 2m + 1 - (\alpha/2)]^2 \right].$$

The actual amount of literature on **Birman–Hardy–Rellich-type inequalities** is enormous: See, e.g., work by **Barbatis, Davies, Edmunds, Evans, Filippas, Gesztesy, Ghoussoub, Herbst, Hinz, Littlejohn, Machihara, Michael, Moradifam, Musina, Ozawa, Pang, Ruzhansky, Simon, Suragan, Tertikas, Wadade, Yafaev,** etc., to name just a few.

Power Weights and Logarithmic Refinements:

Basic setup: Introduce **iterated logarithms** $\ln_j(\,\cdot\,)$, $j \in \mathbb{N}$,

$$\ln_1(\,\cdot\,) = \ln(\,\cdot\,), \quad \ln_2(\,\cdot\,) = \ln(\ln_1(\,\cdot\,)), \quad \ln_{j+1}(\,\cdot\,) = \ln(\ln_j(\,\cdot\,)), \quad j \in \mathbb{N},$$

and **iterated exponentials,**

$$e_0 = 0, \quad e_1 = e^{e_0} = 1, \quad e_2 = e^{e_1}, \quad e_{j+1} = e^{e_j}, \quad j \in \mathbb{N}_0.$$

Given $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, we define the constants

$$A(m,\alpha) = \prod_{j=1}^{m} \left(\frac{2j-1-\alpha}{2}\right)^2, \;\; B(m,\alpha) = \frac{1}{4^m} \sum_{k=1}^{m} \prod_{j=1, j\neq k}^{m} (2j-1-\alpha)^2.$$

In addition, we introduce constants $c_\ell(m,\alpha)$ by

$$\sum_{\ell=0}^{2m} c_\ell(m,\alpha)\lambda^\ell = \prod_{j=1}^{m} \left(\lambda^2 - \frac{(2j-1-\alpha)^2}{4}\right),$$

$$c_{2k-1}(m,\alpha) = 0, \; 1 \leq k \leq m.$$

Power Weights & Logarithmic Refinements (contd.):

Theorem 1 (The interval $(0,\rho)$). $m, N \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and $\rho \in (0,\infty)$

(i) If $\gamma \geq e_N \rho$, then for all $f \in C_0^\infty((0,\rho))$,

$$\int_0^\rho dx\, x^\alpha \big|f^{(m)}(x)\big|^2 \geq A(m,\alpha) \int_0^\rho dx\, x^{\alpha-2m} \big|f(x)\big|^2$$

$$+ B(m,\alpha) \sum_{k=1}^{N} \int_0^\rho dx\, x^{\alpha-2m} \prod_{\ell=1}^{k} [\ln_\ell(\gamma/x)]^{-2} \big|f(x)\big|^2$$

$$+ \sum_{j=2}^{m} |c_{2j}(m,\alpha)| A(j,0) \int_0^\rho dx\, x^{\alpha-2m} [\ln(\gamma/x)]^{-2j} \big|f(x)\big|^2$$

$$+ \sum_{j=2}^{m} |c_{2j}(m,\alpha)| B(j,0) \sum_{k=1}^{N-1} \int_0^\rho dx\, x^{\alpha-2m} [\ln(\gamma/x)]^{-2j}$$

$$\times \prod_{\ell=1}^{k} [\ln_{\ell+1}(\gamma/x)]^{-2} \big|f(x)\big|^2.$$

Power Weights & Logarithmic Refinements (contd.):

Theorem 1 (contd.). (The interval (ρ, ∞).) m, N, α, ρ **as before**

(ii) If $\rho \geq e_N \gamma > 0$, then for all $f \in C_0^\infty((\rho, \infty))$,

$$\int_\rho^\infty dx\, x^\alpha \big|f^{(m)}(x)\big|^2 \geq A(m,\alpha) \int_\rho^\infty dx\, x^{\alpha-2m} \big|f(x)\big|^2$$

$$+ B(m,\alpha) \sum_{k=1}^N \int_\rho^\infty dx\, x^{\alpha-2m} \prod_{\ell=1}^k [\ln_\ell(x/\gamma)]^{-2} \big|f(x)\big|^2$$

$$+ \sum_{j=2}^m |c_{2j}(m,\alpha)| A(j,0) \int_\rho^\infty dx\, x^{\alpha-2m} [\ln(x/\gamma)]^{-2j} \big|f(x)\big|^2$$

$$+ \sum_{j=2}^m |c_{2j}(m,\alpha)| B(j,0) \sum_{k=1}^{N-1} \int_\rho^\infty dx\, x^{\alpha-2m} [\ln(x/\gamma)]^{-2j}$$

$$\times \prod_{\ell=1}^k [\ln_{\ell+1}(x/\gamma)]^{-2} \big|f(x)\big|^2.$$

Power Weights & Logarithmic Refinements (contd.):

Theorem 1 (contd.).

(iii) All inequalities are **strict** for $f \not\equiv 0$ on $(0, \rho)$, respectively, (ρ, ∞).

(iv) The constants $A(m, \alpha)$, $B(m, \alpha)$ **are sharp**.

Note. (i) The two terms containing $c_{2j}(m, \alpha)$ are new.

(ii) For $m \geq 2$ these inequalities are **new** since $\alpha \in \mathbb{R}$ is **unrestricted** and the **sharp conditions** $\gamma \geq e_N \rho$, respectively, $\rho \geq e_N \gamma$ are imposed. Before it was unknown how large γ had to be; only the existence of sufficiently large γ had been established.

These new results became possible after combining an old trick by **P. Hartman (1948)** with an approach by **E. Müller-Pfeiffer (1981)** as we will indicate next.

A New Proof Strategy:

For simplicity we consider only the interval $(0, \rho)$, $0 < \rho < \infty$, and introduce the following **change of variables** and **transformation of functions**:

$$x = \gamma y \in (0, \rho), \quad y = e^{-t} \in (0, \rho/\gamma), \quad t \in (e_{N-1}, \infty), \quad \gamma \geq e_N \rho$$

$$f(x) = g(y) = g(e^{-t}) = e^{-\{m-[(1+\alpha)/2]\}t} u(t),$$

$$f \in C_0^\infty((0, \rho)), \quad u \in C_0^\infty((e_{N-1}, \infty)),$$

$$\sum_{k=1}^{N} \prod_{\ell=1}^{k} [\ln_\ell(\gamma/x)]^{-2} = t^{-2} + t^{-2} \sum_{k=1}^{N-1} \prod_{\ell=1}^{k} [\ln_\ell(t)]^{-2}. \quad \boxed{!!!!!!!!}$$

A New Proof Strategy (contd.):

We want to prove:

$$\int_0^\rho dx\, x^\alpha \left|f^{(m)}(x)\right|^2 \geq A(m,\alpha) \int_0^\rho dx\, x^{\alpha-2m} \left|f(x)\right|^2$$

$$+ B(m,\alpha) \sum_{k=1}^{N} \int_0^\rho dx\, x^{\alpha-2m} \prod_{\ell=1}^{k} [\ln_\ell(\gamma/x)]^{-2} \left|f(x)\right|^2$$

$$+ \sum_{j=2}^{m} |c_{2j}(m,\alpha)| A(j,0) \int_0^\rho dx\, x^{\alpha-2m} [\ln(\gamma/x)]^{-2j} \left|f(x)\right|^2$$

$$+ \sum_{j=2}^{m} |c_{2j}(m,\alpha)| B(j,0) \sum_{k=1}^{N-1} \int_0^\rho dx\, x^{\alpha-2m} [\ln(\gamma/x)]^{-2j}$$

$$\times \prod_{\ell=1}^{k} [\ln_{\ell+1}(\gamma/x)]^{-2} \left|f(x)\right|^2.$$

A New Proof Strategy (contd.):

Thus, for $f \in C_0^\infty((0,\rho))$, $u \in C_0^\infty((e_{N-1},\infty))$,

$$\int_0^\rho dx \Bigg\{ x^\alpha \left|f^{(m)}(x)\right|^2 - A(m,\alpha)x^{\alpha-2m}|f(x)|^2$$

$$- B(m,\alpha)x^{\alpha-2m}\sum_{k=1}^{N}\prod_{\ell=1}^{k}[\ln_\ell(\gamma/x)]^{-2}|f(x)|^2$$

$$- \sum_{j=2}^{m}|c_{2j}(m,\alpha)|A(j,0)x^{\alpha-2m}[\ln(\gamma/x)]^{-2j}|f(x)|^2$$

$$- \sum_{j=2}^{m}|c_{2j}(m,\alpha)|B(j,0)x^{\alpha-2m}[\ln(\gamma/x)]^{-2j}$$

$$\times \sum_{k=1}^{N-1}\prod_{\ell=1}^{k}[\ln_{\ell+1}(\gamma/x)]^{-2}|f(x)|^2 \Bigg\}$$

A New Proof Strategy (contd.):

$$
\begin{aligned}
=&\gamma^{\alpha-2m+1}\sum_{j=1}^{m}|c_{2j}(m,\alpha)|\Big\{\int_{e_{N-1}}^{\infty}dt\,\left|u^{(j)}(t)\right|^2\\
&-A(j,0)\int_{e_{N-1}}^{\infty}dt\,t^{-2j}|u(t)|^2\\
&-B(j,0)\sum_{k=1}^{N-1}\int_{e_{N-1}}^{\infty}dt\,t^{-2j}\prod_{\ell=1}^{k}[\ln_\ell(t)]^{-2}|u(t)|^2\Big\}.
\end{aligned}
$$

a significant simplification!

A New Proof Strategy (contd.):

Now consider $N = 1$:

$$\int_0^{\rho} dx \left\{ x^{\alpha} \left| f^{(m)}(x) \right|^2 - A(m,\alpha) x^{\alpha-2m} |f(x)|^2 \right.$$

$$- B(m,\alpha) x^{\alpha-2m} [\ln(\gamma/x)]^{-2} |f(x)|^2$$

$$\left. - \sum_{j=2}^{m} |c_{2j}(m,\alpha)| A(j,0) x^{\alpha-2m} [\ln(\gamma/x)]^{-2j} |f(x)|^2 \right\}$$

$$= \gamma^{\alpha-2m+1} \sum_{j=1}^{m} |c_{2j}(m,\alpha)| \left\{ \int_0^{\infty} dt \left| u^{(j)}(t) \right|^2 \right.$$

$$\left. - A(j,0) \int_0^{\infty} dt\, t^{-2j} |u(t)|^2 \right\}$$

$$\geq 0, \quad \boxed{\textbf{by Birman's sequence of inequalities (1961)}}$$

At this point one can use induction upon N. This proves these inequalities.

Davies-Hinz Inequalities with log refinements:

Theorem 2. On $B(0,\rho)\backslash\{0\} \subseteq \mathbb{R}^n$. $m, N \in \mathbb{N}, \rho \in (0,\infty)$.

(i) For $\gamma \geq e_N \rho$, $\alpha \in [4m-n, n]$, and $f \in C_0^\infty(B(0,\rho)\backslash\{0\})$:

$$\int_{B(0,\rho)} dx\, |x|^\alpha \big|\big((-\Delta)^m f\big)(x)\big|^2 \geq A(2m,n,\alpha) \int_{B(0,\rho)} dx\, |x|^{\alpha-4m} |f(x)|^2$$

$$+ B(2m,n,\alpha) \sum_{k=1}^{N} \int_{B(0,\rho)} dx\, |x|^{\alpha-4m} \prod_{\ell=1}^{k} [\ln_\ell(\gamma/|x|)]^{-2} |f(x)|^2$$

$$+ \sum_{j=2}^{2m} |c_{2j}(m,\alpha,n)| A(j,0) \int_{B(0,\rho)} dx\, |x|^{\alpha-4m} [\ln(\gamma/|x|)]^{-2j} |f(x)|^2$$

$$+ \sum_{j=2}^{2m} |c_{2j}(m,\alpha,n)| B(j,0) \sum_{k=1}^{N-1} \int_{B(0,\rho)} dx\, |x|^{\alpha-4m} [\ln(\gamma/|x|)]^{-2j}$$

$$\times \prod_{\ell=1}^{k} [\ln_{\ell+1}(\gamma/|x|)]^{-2} |f(x)|^2.$$

Log Refined Davies-Hinz Inequalities (contd.):

Theorem 2. (contd.).

(ii) For $\gamma \geq e_N \rho,\ \alpha \in [4m+2-n, n+2],\ f \in C_0^\infty(B(0,\rho)\backslash\{0\})$:,

$$\int_{B(0,\rho)} dx\, |x|^\alpha \big|(\nabla(-\Delta)^m f)(x)\big|^2 \geq A(2m+1,n,\alpha) \int_{B(0,\rho)} dx\, |x|^{\alpha-4m-2} |f(x)|^2$$

$$+ B(2m+1,n,\alpha) \sum_{k=1}^{N} \int_{B(0,\rho)} dx\, |x|^{\alpha-4m-2} \prod_{\ell=1}^{k} [\ln_\ell(\gamma/|x|)]^{-2} |f(x)|^2$$

$$+ \sum_{j=2}^{2m+1} |\tilde{c}_{2j}(m,\alpha,n)| A(j,0) \int_{B(0,\rho)} dx\, |x|^{\alpha-4m-2} [\ln(\gamma/|x|)]^{-2j} |f(x)|^2$$

$$+ \sum_{j=2}^{2m+1} |\tilde{c}_{2j}(m,\alpha,n)| B(j,0) \sum_{k=1}^{N-1} \int_{B(0,\rho)} dx\, |x|^{\alpha-4m-2} [\ln(\gamma/|x|)]^{-2j}$$

$$\times \prod_{\ell=1}^{k} [\ln_{\ell+1}(\gamma/|x|)]^{-2} |f(x)|^2.$$

Log Refined Davies-Hinz Inequalities (contd.):

Theorem 2. (contd.).

where

$$A(2m,n,\alpha) = \left[\prod_{j=0}^{m-1} [(n/2) + 2j - 2m + (\alpha/2)]^2\right] \times \left[\prod_{k=1}^{m} [(n/2) - 2k + 2m - (\alpha/2)]^2\right],$$

$$A(2m+1,n,\alpha) = \left[\prod_{j=0}^{m} [(n/2) + 2j - 2m - 1 + (\alpha/2)]^2\right] \times \left[\prod_{k=1}^{m} [(n/2) - 2k + 2m + 1 - (\alpha/2)]^2\right],$$

Log Refined Davies-Hinz Inequalities (contd.):

Theorem 2. (contd.).

$$B(2m, n, \alpha) = 4^{-1} A(2m, n, \alpha) \times \left[\sum_{j=0}^{m-1} [(n/2) + 2j - 2m + (\alpha/2)]^{-2} + \sum_{k=1}^{m} [(n/2) - 2k + 2m - (\alpha/2)]^{-2} \right],$$

$$B(2m+1, n, \alpha) = 4^{-1} A(2m+1, n, \alpha) \times \left[\sum_{j=0}^{m} [(n/2) + 2j - 2m - 1 + (\alpha/2)]^{-2} + \sum_{k=1}^{m} [(n/2) - 2k + 2m + 1 - (\alpha/2)]^{-2} \right].$$

Log Refined Davies-Hinz Inequalities (contd.):

Theorem 2. (contd.).

and the constants $c_j(m, \alpha, n)$ and $\tilde{c}_j(m, \alpha, n)$ are defined by

$$\sum_{j=0}^{4m} c_j(m, \alpha, n)\lambda^j = \prod_{\ell=0}^{m-1} \left[\left(\lambda^2 - (2\ell - K)^2 \right) \right] \times \prod_{i=0}^{m} \left[\left(\lambda^2 - (n - 2i + K)^2 \right) \right],$$

and

$$\sum_{j=0}^{4m+2} \tilde{c}_j(m, \alpha, n)\lambda^j = \prod_{\ell=0}^{m-1} \left[\left(\lambda^2 - (2\ell - \tilde{K})^2 \right) \right] \times \prod_{i=0}^{m} \left[\left(\lambda^2 - (n - 2i + \tilde{K})^2 \right) \right],$$

Log Refined Davies-Hinz Inequalities (contd.):

Theorem 2. (contd.).

$$K = 2m - (\alpha/2) - (n/2),$$
$$\tilde{K} = 2m + 1 - (\alpha/2) - (n/2).$$

Domination of Laplacian over radial Laplacian:

Theorem 3 (Powers of Laplacian dominated over powers of radial Laplacian)

Let $m \in \mathbb{N}$ and $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$. Then,

$$\int_{\mathbb{R}^n} d^n x\, |x|^\alpha \big| \big((-\Delta)^m f \big)(x) \big|^2 \geq \int_{\mathbb{R}^n} d^n x\, |x|^\alpha \big| \big((-\Delta_r)^m f \big)(x) \big|^2$$

$$\alpha \in [4m - n, n],$$

$$\int_{\mathbb{R}^n} d^n x\, |x|^\alpha \big| \big(\nabla (-\Delta)^m f \big)(x) \big|^2 \geq \int_{\mathbb{R}^n} d^n x\, |x|^\alpha \big| \big((\partial/\partial r)(-\Delta_r)^m f \big)(x) \big|^2$$

$$\alpha \in [4m + 2 - n, n + 2].$$

Domination:

Theorem 3 (contd.)

where

$$-\Delta_r = -\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} = -r^{1-n}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\right).$$

Birman Inequalities for Radial Laplacians:

Theorem 4. (The interval $(0,\rho)$. $m, N \in \mathbb{N}$, $\alpha, n \in \mathbb{R}$, $\rho \in (0,\infty)$)

If $\gamma \geq e_N \rho$, then for all $f \in C_0^\infty((0,\rho))$,

$$\int_0^\rho r^{n-1} dr\, r^\alpha \big|\big((-\Delta_r)^m f\big)(r)\big|^2 \geq A(2m,n,\alpha) \int_0^\rho r^{n-1} dr\, r^{\alpha-4m} |f(r)|^2$$

$$+ B(2m,n,\alpha) \sum_{k=1}^{N} \int_0^\rho r^{n-1} dr\, r^{\alpha-4m} \prod_{\ell=1}^{k} [\ln_\ell(\gamma/r)]^{-2} |f(r)|^2$$

$$+ \sum_{j=2}^{2m} |c_{2j}(m,\alpha,n)| A(j,0) \int_0^\rho r^{n-1} dr\, r^{\alpha-4m} [\ln(\gamma/r)]^{-2j} |f(r)|^2$$

$$+ \sum_{j=2}^{2m} |c_{2j}(m,\alpha,n)| B(j,0) \sum_{k=1}^{N-1} \int_0^\rho r^{n-1} dr\, r^{\alpha-4m} [\ln(\gamma/r)]^{-2j}$$

$$\times \prod_{\ell=1}^{k} [\ln_{\ell+1}(\gamma/r)]^{-2} |f(r)|^2.$$

Birman Inequalities for Radial Laplacians:

Theorem 4 (contd.).

$$\int_0^\rho r^{n-1}dr\, r^{\alpha}\big|\big((d/dr)(-\Delta_r)^m f\big)(r)\big|^2$$

$$\geq A(2m+1,n,\alpha)\int_0^\rho r^{n-1}dr\, r^{\alpha-4m-2}|f(r)|^2$$

$$+ B(2m+1,n,\alpha)\sum_{k=1}^{N}\int_0^\rho r^{n-1}dr\, r^{\alpha-4m-2}\prod_{\ell=1}^{k}[\ln_\ell(\gamma/r)]^{-2}|f(r)|^2$$

$$+ \sum_{j=2}^{2m+1}|\tilde{c}_{2j}(m,\alpha,n)|A(j,0)\int_0^\rho r^{n-1}dr\, r^{\alpha-4m-2}[\ln(\gamma/r)]^{-2j}|f(r)|^2$$

$$+ \sum_{j=2}^{2m+1}|\tilde{c}_{2j}(m,\alpha,n)|B(j,0)\sum_{k=1}^{N-1}\int_0^\rho r^{n-1}dr\, r^{\alpha-4m-2}[\ln(\gamma/r)]^{-2j}$$

$$\times\prod_{\ell=1}^{k}[\ln_{\ell+1}(\gamma/r)]^{-2}|f(r)|^2.$$