Spectral Theory of Exceptional Hermite polynomials

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From Operator Theory to Orthogonal Polynomials, Combinatorics, and Number Theory
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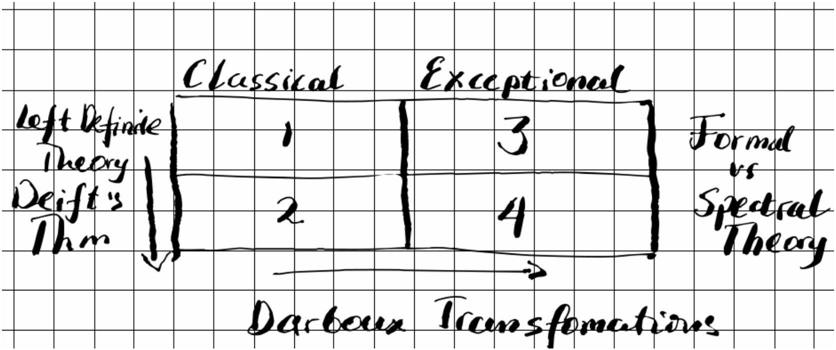
Acknowledgements

- Joint work: D. Gómez-Ullate, Y. Grandati, and R.M.
  ”Spectral Theory of Exceptional Hermite Polynomials”, 2021.
- In the footsteps of: J. Kelly, C. Liaw, L. Littlejohn
  “Spectral analysis for the exceptional X_m-Jacobi equation”, 2015
- With special thanks to F. Gesztesy, “Spectral Characterization of the Double Commutator Method”, 1992

Outline: focus on Hermite polynomials.



Hermite functions and the harmonic oscillator

- Define $h_n(x), \tilde{h}_n(x)$ as repeated derivatives of e^{-x^2}, e^{x^2}.

In[6]:=
```
Table[D[Exp[x^2], {x, n}], {n, 0, 5}] // Factor
```

Out[6]= $\{e^{x^2},\ 2 e^{x^2} x,\ 2 e^{x^2} (1+2x^2),\ 4 e^{x^2} x (3+2x^2),\ 4 e^{x^2} (3+12x^2+4x^4),\ 8 e^{x^2} x (15+20x^2+4x^4)\}$

In[7]:=
```
Table[D[Exp[-x^2], {x, n}], {n, 0, 5}] // Factor
```

Out[7]= $\{e^{-x^2},\ -2 e^{-x^2} x,\ 2 e^{-x^2} (-1+2x^2),\ -4 e^{-x^2} x (-3+2x^2),\ 4 e^{-x^2} (3-12x^2+4x^4),\ -8 e^{-x^2} x (15-20x^2+4x^4)\}$

- Eigenfunctions of harmonic oscillator $\theta = -D_x^2 + x^2 - 1$

$$\psi_n = \begin{cases} e^{-\frac{x^2}{2}} h_n(x) & \text{if } n \geq 0 \\ e^{\frac{x^2}{2}} \tilde{h}_{-n-1}(x) & \text{if } n < 0 \end{cases}$$

- Formal spectrum: $\theta\psi_n = 2n\psi_n,\ n \in \mathbb{Z}$

In[1]:=
```
Join[Table[Exp[-x^2/2] D[Exp[x^2], {x, n}], {n, 3, 0, -1}], Table[(-1)^n Exp[x^2/2] D[Exp[-x^2], {x, n}], {n, 
  Factor
Map[(-D[#, x, x] + (x^2 - 1) #)/# &, %] // Factor
```

Out[1]= $\{4 e^{\frac{x^2}{2}} x (3+2x^2),\ 2 e^{\frac{x^2}{2}} (1+2x^2),\ 2 e^{\frac{x^2}{2}} x,\ e^{\frac{x^2}{2}},\ e^{-\frac{x^2}{2}},\ 2 e^{-\frac{x^2}{2}} x,\ 2 e^{-\frac{x^2}{2}} (-1+2x^2),\ 4 e^{-\frac{x^2}{2}} x (-3+2x^2)\}$

Out[2]= $\{-8, -6, -4, -2, 0, 2, 4, 6\}$

- Maya diagram: L^2 eigenfunctions correspond to $\circ$ asymptotics.

n	..	-5	-4	-3	-2	-1	0	1	2	3	4	...
	...	x	x	x	x	x	o	o	o	o	o	...
λ	...	-10	-8	-6	-4	-2	0	2	4	6	8	...

Ladder operators

- Hermite differential equation: $(\tau + 2n)h_n = 0$, where

$$\tau = -e^{\frac{x^2}{2}}\theta e^{-\frac{x^2}{2}} = D_x^2 - 2xD_x$$

- Lowering/Raising operators: $\alpha = D_x,\ \beta = e^{x^2}D_x e^{-x^2} = D_x - 2x$
- Factorizations: $\tau = \beta\alpha, \quad \tau - 2 = \alpha\beta$.
- Intertwining relations: $\alpha\tau = (\tau - 2)\alpha,\ \tau\beta = \beta(\tau - 2)$.
- Consequence: $\alpha h_n = 2nh_{n-1},\ \beta h_n = -h_{n+1}$.

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APPLICATIONS OF A COMMUTATION FORMULA

P. A. DEIFT

We have the following fundamental result of von Neumann (for a proof see e.g., Riesz-Sz.-Nagy [26], Kato [27], Reed-Simon [22]).

THEOREM 1. *Let A be a (densely defined) closed linear operator from a Hilbert space $\mathscr{H}_1$ to a Hilbert space $\mathscr{H}_2$. Then A^*A, defined naturally, is a (densely defined) positive, self-adjoint operator in $\mathscr{H}_1$. Moreover*

$$D(A) = Q(A^*A) = D((A^*A)^{1/2})$$

where $Q(A^*A)$ is the form domain of A^*A. □

We prove the following theorem

THEOREM 2 (Commutation). *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Hilbert spaces. Suppose that*

(i) *A is a bounded, linear operator from $\mathscr{H}_1$ to $\mathscr{H}_2$ and that B is a bounded linear operator from $\mathscr{H}_2$ to $\mathscr{H}_1$,*
or that

(ii) *A is a (densely defined) closed linear operator from $\mathscr{H}_1$ to $\mathscr{H}_2$ and that $B = A^*$.*

Let $S = AB$ and let $T = BA$, defined everywhere on $\mathscr{H}_2$, $\mathscr{H}_1$ respectively in case (i), *and defined naturally (as self-adjoint operators) in case* (ii).

Then the spectra of S and T are identical away from zero. Moreover, if $-\lambda \neq 0$ is an eigenvalue of S (respectively T) then $-\lambda$ is an eigenvalue of T (respectively S) and B (respectively A) is a surjection of $N(S + \lambda)$ (respectively $N(T + \lambda)$) onto $N(T + \lambda)$ (respectively $N(S + \lambda)$). In particular $N(T + \lambda)$ and $N(S + \lambda)$ have the same dimension.

Classical Hermite Polynomials: spectral theory

- Hermite diffeq: $(\tau + 2n)h_n = 0$, where $\tau = -e^{\frac{x^2}{2}}\theta e^{-\frac{x^2}{2}} = D_x^2 - 2xD_x$
- Reformulate as SLP: $\left(-e^{-x^2}y'\right)' = \lambda e^{-x^2}y,\ x \in (-\infty, \infty)$
  Note: limit point, so no need for boundary conditions.
- Orthogonality of eigenpolynomials wrt $W(x) = e^{-x^2},\ x \in \mathbb{R}$
- Extend $-\tau$ to unbounded self-adjoint T on $\mathcal{H} = \mathrm{L}^2(\mathbb{R}, W)$

$$\mathrm{Dom}(T) = \{f \in \mathcal{H} \colon f' \in \mathrm{AC}_{\mathrm{loc}},\ \tau f \in \mathcal{H}\}$$

- **Proposition:** $\sigma(T) = \{2n : n \in \mathbb{N}_0\}$
  Operator spectrum recovered from formal spectrum; i.e., no "exotic", non-polynomial eigenfunctions
- Proof. Extend lowering operator $\alpha = D_x$ to closed, densely defined A with left-definite domain. Observe that $T = A^*A$ and that all eigenfunctions are in $D(A)$.
- By inspection, $T \geq 0$ and $2\mathbb{N}_0 \subset \sigma(T)$.
- Almost isospectral $\sigma(T + 2) = \sigma(T) \setminus \{0\}$
- Note: this gives a spectral-theoretic proof of the completeness of the Hermite polynomials in $\mathcal{H}$.

Exceptional Hermite polynomials: formal theory

- Exceptional OP: eigenpolynomials of a 2nd order SLP
  Allow a finite number of missing "exceptional" degrees.
- Exceptional Hermite polynomials: missing degrees $K = \{k_1, \ldots, k_N\}$
  Partition Nth triangular number into N distinct parts.

$$k_1 + \cdots + k_N = \frac{1}{2}N(N+1)/2, \quad 0 \leq k_1 < \cdots < k_N$$

$$\eta_K = C_K \,\mathrm{Wr}[h_{k_1}, \ldots, h_{k_N}] = 2^N x^N + \text{lower degree terms}$$

$$h_{K,n} = C_{K,n} \,\mathrm{Wr}[h_{k_1}, \ldots, h_{k_N}, h_n] = 2^n x^n + \text{l.d.t.}, \quad n \notin K,\ n \geq 0$$

- Exceptional Hermite functions: $\psi_{K,n} = \begin{cases} e^{-\frac{x^2}{2}} \frac{h_{K,n}}{\eta_K}, & n \notin K,\ n \geq 0 \\ e^{\frac{x^2}{2}} \frac{\tilde{h}_{K,n}}{\eta_K}, & \text{otherwise} \end{cases}$
- Darboux-Crum formula: $\theta_K = -D_{xx} + x^2 + 2N - 1 - 2D_{xx} \log \eta_K$
- Preserves formal spectrum: $\theta_K \psi_{K,n} = 2n\, \psi_{K,n},\ n \in \mathbb{Z}$.
- X-operator: $\tau_K = -(e^{\frac{x^2}{2}} \eta_K)\theta_K \frac{e^{-\frac{x^2}{2}}}{\eta_K} = D_{xx} - 2xD_x - 2\frac{\eta_K{}'}{\eta_K} D_x + \left(\frac{\eta_K{}''}{\eta_K} + 2x\frac{\eta_K{}'}{\eta_K}\right)$
- Exceptional hermite diffeq: $(\tau_K + 2n)h_{K,n} = 0,\ n \notin K,\ n \geq 0$.

Example

- Missing degrees: $6 = 0 + 1 + 5$
- $\eta_K = \frac{1}{160}\,\mathrm{Wr}[h_0, h_1, h_5] = 8x^3 - 12x$
- Exceptional operator and Hermite functions:

```
In[91]:= H[$tau, f[x]] // Map[Factor, #] &
         Table[Exp[-x^2/2] hh[$K, n] / $tau, {n, -2, 6}] // Factor
         Map[H[$tau, #]/# &, %] // Factor
```

Out[91]= $\left(5 + x^2\right) f[x] + \dfrac{6\left(3 + 4x^4\right) f[x]}{x^2\left(-3 + 2x^2\right)^2} - f''[x]$

Out[92]= $\Big\{\dfrac{8\, e^{\frac{x^2}{2}} x^2\left(-21 + 4x^4\right)}{-3 + 2x^2}, \dfrac{2\, e^{\frac{x^2}{2}}\left(3 - 18x^2 - 12x^4 + 8x^6\right)}{x\left(-3 + 2x^2\right)},$
$\dfrac{e^{\frac{x^2}{2}} x^2\left(-5 + 2x^2\right)}{5\left(-3 + 2x^2\right)}, \dfrac{e^{\frac{x^2}{2}}\left(3 - 12x^2 + 4x^4\right)}{16\, x\left(-3 + 2x^2\right)}, -\dfrac{24\, e^{-\frac{x^2}{2}}\left(-1 + 2x^2\right)}{x\left(-3 + 2x^2\right)}, -\dfrac{192\, e^{-\frac{x^2}{2}} x^2}{-3 + 2x^2},$
$-\dfrac{96\, e^{-\frac{x^2}{2}}\left(3 + 4x^4\right)}{x\left(-3 + 2x^2\right)}, \dfrac{e^{\frac{x^2}{2}}}{320\, x\left(-3 + 2x^2\right)}, \dfrac{480\, e^{-\frac{x^2}{2}}\left(9 + 18x^2 - 12x^4 + 8x^6\right)}{x\left(-3 + 2x^2\right)}\Big\}$

Out[93]= {-4, -2, 0, 2, 4, 6, 8, 10, 12}

- Formal spectrum
  Maya diagram

n	…	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	…	
	…	X	X	X	X	X	O	O	O	O	O	O	O	…	Classical
	…	X	X	X	X	X	X	X	O	O	O	X	O		Exceptional
λ	…	-10	-8	-6	-4	-2	0	2	4	6	8	10	12	…	

Example

- Missing degrees: $10 = 0 + 1 + 4 + 5$
- $\eta_K = \frac{1}{1560} \mathrm{Wr}[h_0, h_1, h_4, h_5] = 16x^4 + 12$
- Exceptional operator and Hermite functions:

```
In[180]:= H[$tau, f[x]] // Map[Factor, #] &
          Table[Exp[-x^2/2] hh[$K, n] / $tau, {n, -2, 6}] // Factor
          Map[H[$tau, #]/# &, %] // Factor
```

Out[180]=

$$\left(7 + x^2\right) f[x] + \frac{32\, x^2 \left(-3 + 2\, x^2\right) \left(3 + 2\, x^2\right) f[x]}{\left(3 + 4\, x^4\right)^2} - f''[x]$$

Out[181]=

$$\left\{ \frac{122\,880\, e^{\frac{x^2}{2}}\, x \left(63 + 84\, x^2 + 48\, x^6 + 16\, x^8\right)}{3 + 4\, x^4}, \frac{61\,440\, e^{\frac{x^2}{2}} \left(15 + 60\, x^2 + 16\, x^6 + 16\, x^8\right)}{3 + 4\, x^4}, \right.$$

$$\frac{384\, e^{\frac{x^2}{2}}\, x \left(15 + 10\, x^2 - 4\, x^4 + 8\, x^6\right)}{3 + 4\, x^4}, \frac{320\, e^{\frac{x^2}{2}} \left(9 + 18\, x^2 - 12\, x^4 + 8\, x^6\right)}{3 + 4\, x^4},$$

$$\frac{1\,474\,560\, e^{-\frac{x^2}{2}} \left(1 + 2\, x^2\right)}{3 + 4\, x^4}, \frac{2\,949\,120\, e^{-\frac{x^2}{2}}\, x \left(3 + 2\, x^2\right)}{3 + 4\, x^4}, \frac{160\, e^{\frac{x^2}{2}}\, x \left(-3 + 2\, x^2\right)}{3 + 4\, x^4},$$

$$\left. \frac{48\, e^{\frac{x^2}{2}} \left(-1 + 2\, x^2\right)}{3 + 4\, x^4}, \frac{29\,491\,200\, e^{-\frac{x^2}{2}} \left(-9 + 18\, x^2 + 12\, x^4 + 8\, x^6\right)}{3 + 4\, x^4} \right\}$$

Out[182]=

{-4, -2, 0, 2, 4, 6, 8, 10, 12}

- Formal spectrum
  Maya diagram

n	..	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	...		
	...	X	X	X	X	X	O	O	O	O	O	O	O	...		Classical
	...	X	X	X	X	X	X	X	O	O	X	X	O			Exceptional
λ	...	-10	-8	-6	-4	-2	0	2	4	6	8	10	12	...		

Darboux transformations: formal theory

- Starting point: Schrodinger operator and seed eigenfunction.

$$\theta = -D_{xx} + U(x), \quad H\phi = \epsilon\phi.$$

- Factorization: $\theta = \alpha^\dagger\alpha + \epsilon$ where

$$\alpha = -D_x + w(x), \quad \alpha^\dagger = D_x + w(x), \quad w = \phi'/\phi,\ w' + w^2 = U - \epsilon.$$

- Partner operator: $\widehat{\theta} = \alpha\alpha^\dagger + \epsilon = \theta - 2w'$.
- Intertwining relations: $\widehat{\theta}\alpha = \alpha\theta$
- Intertwiner: $\alpha : \ker(\theta - \lambda) =: \mathcal{E}_\lambda(\theta) \to \mathcal{E}_\lambda(\widehat{\theta})$

$$\theta\phi = \lambda\phi,\ \widehat{\phi} := \alpha\phi \quad \Rightarrow \quad \widehat{\theta}\widehat{\phi} = \widehat{\theta}\alpha\phi = \alpha\theta\phi = \lambda\widehat{\phi}.$$

Factorization chains: Darboux-Crum formula

- Starting operator $\theta = -D_{xx} + U(x)$
  Seed eigenfunctions $\phi_1, \ldots, \phi_N$ s.t. $\theta\phi_i = \epsilon_i\phi_i$.
- Dressing chain: $\theta = \theta_0 \to \theta_1 \to \cdots \to \theta_N$ with

$$\theta_i = \theta_0 - 2D_{xx} \log \eta_i, \quad \eta_i = \mathrm{Wr}[\phi_1, \ldots, \phi_i].$$

- Factorization chain: $\theta_{i-1} = \alpha_i^\dagger \alpha_i + \epsilon_i,\ \theta_i = \alpha_i \alpha_i^\dagger + \epsilon_i, \quad i = 1, \ldots N$

$$\alpha_i = -D_x + w_i(x), \quad w_i = \frac{\eta_i'}{\eta_i} - \frac{\eta_{i-1}'}{\eta_{i-1}}.$$

- Intertwining relations: $\theta_i \hat{\alpha}_i = \hat{\alpha}_i \theta_0$ where $\hat{\alpha}_i = \alpha_i \cdots \alpha_1$ with

$$\hat{\alpha}_i f = \pm \frac{\mathrm{Wr}[\phi_1, \ldots, \phi_i, f]}{\mathrm{Wr}[\phi_1, \ldots, \phi_i]}.$$

Hermite factorization chains

- Fix partition of N, indices $k_1 < \cdots < k_N$. Set $\eta_i = \mathrm{Wr}[H_{k_1}, \ldots, H_{k_i}]$.
- Define weights $W_i = e^{-x^2}/\eta_i(x)^2$ and operators

$$\tau_i = D_{xx} - 2xD_x - 2\frac{\eta_i'}{\eta_i}D_x + \left(\frac{\eta_i''}{\eta_i} + 2x\frac{\eta_i'}{\eta_i}\right) + 2k_i$$

  with $W_i\tau_i y = (W_i y')' + R_i y$ formally symmetric.
- Factorization chain: $\tau_{i-1} = \beta_i\alpha_i - 2i,\ \tau_i = \alpha_i\beta_i - 2i$ where

$$\alpha_i y = \mathrm{Wr}[\eta_i, y]/\eta_{i-1}, \quad \beta_i y = e^{-x^2}\,\mathrm{Wr}[e^{x^2}\eta_{i-1}, y]/\eta_i$$

- **Proposition.** The formal spectrum of τ_i is the Maya diagram obtained from the trivial Maya Diagram by flipping asymptotics at sites $k_1, \ldots, k_i$.
  Note: in the polynomial gauge, $\circ$ indicates a polynomial eigenfunction and $\times$ indicates a polynomial times e^{x^2}.

Regularity

- Formal inner product: $\langle f, g\rangle = \int_C f(x)\overline{g(x)}\frac{e^{-x^2}}{\eta_K(x)^2}dx$
  where C is a contour from $-\infty$ to ∞, avoiding zeros of $\eta_K(x)$.
- Formal orthogonality and norming constants:

$$\langle H_{K,m} H_{K,n}\rangle = \delta_{m,n}\frac{2^n n!\sqrt{\pi}}{2^N \pi_K(n)}, \quad \pi_K(n) = (n-k_1)\cdots(n-k_N)$$

- **Theorem (Krein-Adler):** The following are equivalent:
  - $\eta_K(x) = C_K \operatorname{Wr}[h_{k_1}, \dots, h_{k_N}]$ has no real zeros
  - $U_K(x) = x^2 + 2N - 1 - 2D_{xx}\log\eta_K(x)$ is non-singular on $\mathbb{R}$
  - $\pi_K(n) \geq 0$ for $n \in \mathbb{N}_0$
  - The finite blocks of the Maya diagram $\{\dots, -2, -1\} \cup K$ are even
- **X-Hermite Spectral Theorem:** Suppose that K satisfies the Krein-Adler conditions. Then τ_K is a non-singular differential expression that can be extended to an unbounded, self-adjoint operator on $\mathcal{H}_K = \mathrm{L}^2(\mathbb{R}, W_K)$. The L^2 spectrum consists of the classical spectrum minus the eigenvalues corresponding to the exceptional degrees.

Regular factorization chains

- **Lemma.** Every regular τ_K admits a factorization chain to the classical τ consisting of regular intermediaries.
- Proof by example:

															N
...	X	X	X	O	O	X	X	O	X	X	O	..	.		10
...	X	X	X	X	O	X	X	O	X	X	O	..	.		6
...	X	X	X	X	X	X	X	O	X	X	O	..	.		2
...	X	X	X	X	X	X	X	X	X	X	O	.	..		0

- Proof of Theorem: Deift's theorems justifies formal DT if both operators are regular. Use Lemma to apply Deift's theorem recursively.
- **Corollary:** Suppose that K satisfies the Krein-Adler conditions. Then $W_K = e^{-x^2}/\eta_K(x)^2,\ x \in \mathbb{R}$ is a non-singular weight and exceptional Hermite polynomials $H_{K,n},\ n \notin K$ are a complete, orthogonal basis in $\mathcal{H}_K = \mathrm{L}^2(\mathbb{R}, W_K)$.

Thank you for your attention.
Thank you Lance!