Optimality of the Birman–Hardy–Rellich Inequalities

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Based on joint work with

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Outline

1. Introduction: Background

In 1961, **M. Š. Birman** established the following sequence of integral inequalities:

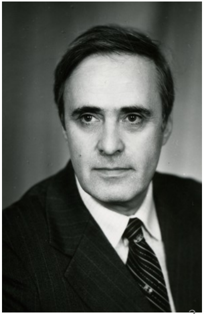
Theorem 1.1 (Birman's Sequence of Inequalities, 1961)

Let $m \in \mathbb{N}$. Then for all $f \in C_0^m((0,\infty))$,

$$\int_0^\infty dx\, \left|f^{(m)}(x)\right|^2 \geq \frac{[(2m-1)!!]^2}{2^{2m}} \int_0^\infty dx\, x^{-2m} |f(x)|^2. \qquad (I_m)$$

In particular, (I_1) is the classical Hardy inequality and (I_2) is the one-dimensional Rellich inequality.

Since the establishment of (I_m),
a great amount of research has been dedicated to improving them, such as: **extending to $p \in [1,\infty)$, optimal function spaces, vector-valued contexts, multi-dimensions, weights, logarithmically weaker singular potentials,** etc.



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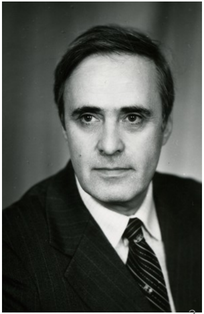

1. Introduction: Literature

This talk begins with a **new proof** of the optimal version of the power-weighted sequence of Birman–Hardy–Rellich inequalities of the form

$$\int_0^\infty dx\, x^\alpha \big|f^{(m)}(x)\big|^2 \geq A(\ell,\alpha) \int_0^\infty dx\, x^{\alpha-2\ell} \big|f^{(m-\ell)}(x)\big|^2,$$
$$m \in \mathbb{N},\ \ell \in \{1,\dots,m\},\ \alpha \in \mathbb{R},\ f \in C_0^\infty((0,\infty)),$$

using the **HMP transformation**. We next investigate an example of a **reverse HMP transformation** and its application to a linear combination of two sharp inequalities. We conclude our talk by briefly discussing multi-dimensions.

Based on:

F. Gesztesy, L. L. Littlejohn, **I. M.**, and **M. M. H. Pang**, *A Sequence of Weighted Birman–Hardy–Rellich Inequalities with Logarithmic Refinements*, Integral Eq. Operator Th., Springer, **94(13)**, (2022).

F. Gesztesy, **I. M.**, and **M. M. H. Pang**, *A New Proof of the Weighted Birman–Hardy–Rellich Inequalities*, (soon to appear).

F. Gesztesy, **I. M.**, and **M. M. H. Pang**, *Optimality of Constants in Power-Weighted Birman–Hardy–Rellich-Type Inequalities with Logarithmic Refinements*, CUBO, A Math. J., **24(1)**, 115-165 (2022).

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During his Presidential Address at the meeting of the London Mathematical Society, on November 8, 1928, G. H. Hardy gave the following quote from Harald Bohr:

"All analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove."

Birman–Hardy–Rellich-type inequalities have numerous applications in:

- Theory of differential equations and approximation theory.
- Sobolev embedding theorems in the context of weighted Lebesgue spaces.
- The study of elliptic and parabolic PDE's, particularly if singular potentials are present.
- Determining lower boundedness of Hamiltonians.
- Self-adjointness and spectral theory problems for second and higher-order differential operators with strongly singular coefficients.

Note: Hardy's multidimensional inequality ($p = 2$) is related to the Heisenberg uncertainty principle in quantum mechanics.

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2. Power-Weighted Birman–Hardy–Rellich Inequalities

Given $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, define

$$A(m,\alpha) = \prod_{j=1}^{m} \left(\frac{2j-1-\alpha}{2} \right)^2.$$

We start by establishing the sequence of power-weighted Birman–Hardy–Rellich inequalities for the case $\ell = m$.

Lemma 2.1

Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$. Then

$$\int_0^\infty dx\, x^\alpha \big| f^{(m)}(x) \big|^2 \geq A(m,\alpha) \int_0^\infty dx\, x^{\alpha-2m} \big| f(x) \big|^2,$$

for all $f \in C_0^\infty((0,\infty))$.

Moreover, the constant $A(m,\alpha)$ is optimal and the inequality is strict, that is, equality holds if and only if $f \equiv 0$.

Proof. Let $C \in (0, \infty)$ and define Q as the operator in $L^2((0,\infty); dx)$ given by

$$Q = \left[(-1)^m \frac{d^m}{dx^m} \left(x^\alpha \frac{d^m}{dx^m} \right) - C x^{\alpha - 2m} \right] \Big|_{C_0^\infty((0,\infty))}.$$

Utilizing

$$\int_a^b dx\, x^\alpha \big| f^{(m)}(x) \big|^2 = (-1)^m \int_a^b dx\, \left(x^\alpha f^{(m)}(x) \right)^{(m)} \overline{f(x)},$$
$$m \in \mathbb{N},\ \alpha \in \mathbb{R},\ f \in C_0^\infty((a,b)),\ 0 \leq a < b \leq \infty,$$

one concludes that

$$(f, Qf)_{L^2((0,\infty))} = \int_0^\infty dx \left\{ x^\alpha \big| f^{(m)}(x) \big|^2 - C x^{\alpha - 2m} \big| f(x) \big|^2 \right\},$$

for all $f \in C_0^\infty((0,\infty))$.

Thus, to establish the inequality and, simultaneously, optimality of the constant $A(m, \alpha)$, we will show that

$$Q \geq 0 \text{ if and only if } C \leq A(m, \alpha).$$

To this end, one introduces the following elementary variable transformation, an extension, and combination, of transformations considered by **Hartman** and **Müller-Pfeiffer**:

Assume temporarily that

$$\alpha \in \mathbb{R} \backslash \{j \,|\, 1 \leq j \leq 2m - 1\}.$$

Given $f \in C_0^\infty((0, \infty))$, consider the transformation

$$\begin{aligned} & x = e^t, \; x \in (0, \infty), \quad dx = e^t dt, \; t \in \mathbb{R}, \\ & f(x) \equiv f(e^t) = e^{[(2m-1-\alpha)/2]t} w(t), \quad w \in C_0^\infty(\mathbb{R}). \end{aligned}$$

2. Power-Weighted Birman–Hardy–Rellich Inequalities

The **Hartman–Müller-Pfeiffer (HMP) transformation** yields

$$\big(x^{\alpha} f^{(m)}(x)\big)^{(m)} = e^{-[(2m+1-\alpha)/2]t} \sum_{\ell=0}^{2m} c_{\ell}(m,\alpha) w^{(\ell)}(t),$$

for appropriate constants $c_{\ell}(m,\alpha)$, $\ell = 0, 1, \dots, 2m$ to be determined next.

The solutions of the differential equation

$$\big(x^{\alpha} f^{(m)}(x)\big)^{(m)} = 0,$$

are linear combinations of the following powers of x:

$$\begin{cases} x^{j}, & j = 0, 1, \dots, m-1, \\ x^{k-\alpha}, & k = m, \dots, 2m-1. \end{cases}$$

One notes that these solutions are linearly independent.

2. Power-Weighted Birman–Hardy–Rellich Inequalities

Computations then show that the solutions of

$$\sum_{\ell=0}^{2m} c_\ell(m,\alpha) w^{(\ell)}(t) = 0,$$

are the functions

$$e^{\pm\frac{1}{2}(2j-1-\alpha)t}, \quad j = 1, \dots, m.$$

The zeros of the **characteristic polynomial** of

$$\sum_{\ell=0}^{2m} c_\ell(m,\alpha) w^{(\ell)}(t) = 0,$$

are thus the constant factors

$$\pm\frac{1}{2}(2j-1-\alpha), \quad j = 1, \dots, m.$$

Hence, the characteristic polynomial is given by

$$P_{m,\alpha}(\lambda) = \sum_{\ell=0}^{2m} c_\ell(m,\alpha)\lambda^\ell = \prod_{j=1}^{m}\left(\lambda^2 - \frac{(2j-1-\alpha)^2}{4}\right).$$

2. Power-Weighted Birman–Hardy–Rellich Inequalities

Thus, the coefficients $c_\ell(m,\alpha)$, $\ell = 0, 1, \dots, 2m$, satisfy the following properties:

(*i*) $c_{2j-1}(m,\alpha) = 0, \quad j = 1, \dots, m;$

(*ii*) $c_{2j}(m,\alpha) = (-1)^{m-j}|c_{2j}(m,\alpha)|, \quad j = 0, 1, \dots, m;$

(*iii*) $|c_0(m,\alpha)| = A(m,\alpha)$

(*iv*) $c_{2m}(m,\alpha) = 1.$

(*v*) Saved for later....

Applying the Hartman–Müller-Pfeiffer transformation to

$$(f, Qf)_{L^2((0,\infty))} = \int_0^\infty dx \left\{ x^\alpha \big| f^{(m)}(x) \big|^2 - C x^{\alpha-2m} \big| f(x) \big|^2 \right\},$$

yields the following result:

$$(f, Qf)_{L^2((0,\infty);dx)} = \int_0^\infty dx\, \overline{f(x)} \Big\{ \Big[(-1)^m \big(x^\alpha f^{(m)}(x)\big)^{(m)} - Cx^{\alpha-2m} f(x)\Big] \Big\}$$

$$= \int_{-\infty}^\infty dt \left\{ e^t \left[(-1)^m e^{-[(2m+1-\alpha)/2]t} \sum_{j=0}^m (-1)^{m-j} |c_{2j}(m,\alpha)| w^{(2j)}(t) \right.\right.$$

$$\left.\left. - Ce^{(\alpha-2m)t+[(2m-1-\alpha)/2]t} w(t) \right] e^{[(2m-1-\alpha)/2]t}\overline{w(t)} \right\}$$

$$= \int_{-\infty}^\infty dt \left\{ \overline{w(t)} \Big[\sum_{j=0}^m (-1)^j |c_{2j}(m,\alpha)| w^{(2j)}(t) - Cw(t) \Big] \right\}.$$

Therefore,

$$Q \geq 0 \text{ in } L^2((0,\infty); dx)$$

if and only if the constant coefficient operator S in $L^2(\mathbb{R}; dt)$, defined by

$$S = \left[\sum_{j=0}^{m} (-1)^j |c_{2j}(m,\alpha)| \frac{d^{2j}}{dt^{2j}} - C \right] \Big|_{C_0^\infty(\mathbb{R})},$$

satisfies

$$S \geq 0 \text{ in } L^2(\mathbb{R}; dt).$$

Invoking the **Fourier transform**, the closure $\overline{S}$ of S in $L^2(\mathbb{R}; dt)$ is unitarily equivalent to the maximally defined operator of multiplication T in $L^2(\mathbb{R}; d\xi)$ given by the following polynomial,

$$(Tv)(\xi) = \sum_{j=1}^{m} |c_{2j}(m,\alpha)|\xi^{2j} v(\xi) + \big(|c_0(m,\alpha)| - C\big) v(\xi),$$

$$v \in \operatorname{dom}(T) = \left\{ u \in L^2(\mathbb{R}; d\xi) \,\middle|\, \int_{-\infty}^{\infty} d\xi\, \xi^{4m} |u(\xi)|^2 < \infty \right\}.$$

Recalling the property

$$|c_0(m,\alpha)| = A(m,\alpha),$$

it follows that T (and hence Q) is nonnegative if and only if $C \leq A(m,\alpha)$.

Moreover, if $C = A(m,\alpha)$ then $T \geq 0$ with trivial nullspace, $\ker(T) = \{0\}$. Thus, the Birman inequality is strict unless $f \equiv 0$.

The case $\alpha \in \{j \,|\, 1 \leq j \leq 2m-1\}$ then follows by taking the limits $\alpha \to k \in \{j \,|\, 1 \leq j \leq 2m-1\}$, noting that $A(m,\alpha)$ and $c_{2j}(m,\alpha)$ are continuous as polynomials in $\alpha \in \mathbb{R}$.

□

Remark 2.2

This proof is significant in that it simultaneously establishes the Birman inequality and optimality of the Birman constant $A(m,\alpha)$.

By replacing f by $f^{(m-\ell)}$, we can extend to all intermediate cases $\ell \in \{1,\dots,m\}$.

Theorem 2.3

One has

$$\int_0^\infty dx\, x^{\alpha}\big|f^{(m)}(x)\big|^2 \geq A(\ell,\alpha)\int_0^\infty dx\, x^{\alpha-2\ell}\big|f^{(m-\ell)}(x)\big|^2,$$
$$m\in\mathbb{N},\ \ell\in\{1,\dots,m\},\ \alpha\in\mathbb{R},\ f\in C_0^\infty((0,\infty)).$$

Moreover, the constants $A(\ell,\alpha)$, for $1\leq\ell\leq m$, $\alpha\in\mathbb{R}\backslash\{2j-1\}_{\ell+1\leq j\leq m}$, in are optimal and the inequality is strict, that is, equality holds if and only if $f\equiv 0$.

3. Reverse HMP Transformation (Rellich)

Given $m \in \mathbb{N}, \alpha \in \mathbb{R}$, let

$$B(m,\alpha) := \frac{1}{4^m} \sum_{k=1}^{m} \prod_{j=1, j\neq k}^{m} (2j-1-\alpha)^2.$$

The following was proven (in much greater generality) using limiting methods:

Theorem 3.1 (Rellich w. Logarithmic Refinement)

Let $\alpha \in \mathbb{R}$ and $\rho, \gamma \in (0,\infty)$ with $\gamma \geq e\rho$. The inequality

$$\int_0^\rho dx\, x^\alpha \big|f''(x)\big|^2 \geq A(2,\alpha) \int_0^\rho dx\, x^{\alpha-4} \big|f(x)\big|^2 + C \int_0^\rho dx\, x^{\alpha-4} [\ln(\gamma/x)]^{-2} \big|f(x)\big|^2$$

holds for all $f \in C_0^\infty((0,\infty))$ if and only if $C \leq B(2,\alpha)$.

3. Reverse HMP Transformation (Rellich)

Recalling the characteristic polynomial from the HMP transformation

$$P_{m,\alpha}(\lambda) = \sum_{\ell=0}^{2m} c_\ell(m,\alpha)\lambda^\ell = \prod_{j=1}^{m} \left(\lambda^2 - \frac{(2j-1-\alpha)^2}{4}\right),$$

we now state the final property of the coefficients $c_\ell(m,\alpha)$:

(v) $|c_2(m,\alpha)| = 4B(m,\alpha)$

Using Theorem 3.1 and a reverse HMP transformation yields the following:

Corollary 3.2

Let $\alpha \in \mathbb{R}, \rho, \gamma \in (0,\infty), \gamma \geq e\rho$, *and* $\tilde{\rho} = \ln(\gamma/\rho)$.

The inequality

$$\int_{\tilde{\rho}}^{\infty} dx \left|f''(x)\right|^2 + 4B(2,\alpha) \int_{\tilde{\rho}}^{\infty} dx \left|f'(x)\right|^2 \geq C \int_{\tilde{\rho}}^{\infty} dx\, x^{-2} \left|f(x)\right|^2$$

holds for all $f \in C_0^\infty((\tilde{\rho},\infty))$ *if and only if* $C \leq B(2,\alpha)$.

3. Reverse HMP Transformation (Rellich)

Proof. The scaling

$$t = \gamma y, \quad dt = \gamma dy, \quad y \in (0, \rho/\gamma), \quad g(y) = h(\gamma y),$$

yields

$$h \in C_0^\infty((0,\rho)) \iff g \in C_0^\infty((0,\rho/\gamma)).$$

Slightly modifying the HMP transformation (with $m = 2$) applied to g

$$y = e^{-x}, \quad dy = -e^{-x} dt, \quad x \in (\tilde{\rho}, \infty),$$
$$g(y) \equiv g(e^{-x}) = e^{-[(3-\alpha)/2]x} f(x),$$

gives

$$g \in C_0^\infty((0,\rho/\gamma)) \iff f \in C_0^\infty((\tilde{\rho}, \infty)).$$

3. Reverse HMP Transformation (Rellich)

Using the **important fact**

$$y^{\alpha-4}|g(y)|^2 = e^x|f(x)|^2, \qquad y^{\alpha-4}[\ln(1/y)]^{-2}|g(y)|^2 = e^x x^{-2}|f(x)|^2,$$

and the properties

$$(iii)\ |c_0(m,\alpha)| = A(m,\alpha), \qquad (v)\ |c_2(m,\alpha)| = 4B(m,\alpha),$$

computation (via **reverse HMP transform**) shows

$$\begin{aligned}\int_{\tilde{\rho}}^{\infty} dx \left\{|f''(x)|^2 + 4B(2,\alpha)|f'(x)|^2 - Cx^{-2}|f(x)|^2\right\}\\ = \gamma^{3-\alpha}\int_0^{\rho} dt \left\{t^{\alpha}|h''(t)|^2 - A(2,\alpha)t^{\alpha-4}|h(t)|^2\right.\\ \left. - Ct^{\alpha-4}[\ln(\gamma/t)]^{-2}|h(t)|^2\right\}.\end{aligned}$$

By Theorem 3.1, the r.h.s. is nonnegative for all $h \in C_0^{\infty}((0,\rho))$ if and only if $C \leq B(2,\alpha)$.

Hence, the l.h.s. is nonnegative for all $f \in C_0^{\infty}((\tilde{\rho},\infty))$ if and only if $C \leq B(2,\alpha)$.

3. Reverse HMP Transformation (Rellich)

Similar methods (involving $[\ln(\gamma/x)]^{-4}$) yield the following:

Corollary 3.3 (Hardy + Rellich Inequality)

Let $\alpha \in \mathbb{R}, \rho, \gamma \in (0, \infty), \gamma \geq e\rho$, *and* $\tilde{\rho} = \ln(\gamma/\rho)$.

For all $f \in C_0^\infty((\tilde{\rho}, \infty))$,

$$\int_{\tilde{\rho}}^{\infty} dx\, |f''(x)|^2 + 4B(2,\alpha) \int_{\tilde{\rho}}^{\infty} dx\, |f'(x)|^2$$
$$\geq B(2,\alpha) \int_{\tilde{\rho}}^{\infty} dx\, x^{-2}|f(x)|^2 + A(2,0) \int_{\tilde{\rho}}^{\infty} dx\, x^{-4}|f(x)|^2.$$

The constant $B(2,\alpha)$ *on the r.h.s. is sharp.*

Remark 3.4

Recall that, in general, the sum of 'optimal' inequalities is not necessarily 'optimal'. This result suggests the possibility of a linear combination of 'optimal' inequalities which is itself still 'optimal'.

4. Multi-dimensions and Distance to the Boundary

Recalling the **radial derivative** ∂_r centered around $x_0 = 0$,

$$\partial_r := |x|^{-1} x \cdot \nabla, \qquad x \in \mathbb{R}^n \backslash \{0\},\ r = |x|,\ n \in \mathbb{N},\ n \geq 2,$$

we also introduce the **radial Laplacian** Δ_r,

$$\Delta_r := r^{1-n} \partial_r r^{n-1} \partial_r = \partial_r^2 + (n-1) r^{-1} \partial_r,$$

noting that the Laplacian Δ, in n-**dimensional polar coordinates**, can be expressed as

$$\Delta = \Delta_r + r^{-2} \Delta_{S^{n-1}},$$

where $\Delta_{S^{n-1}}$ is the **Laplace-Beltrami operator** on the $(n-1)$-sphere $S^{n-1} \subset \mathbb{R}^n$.

We will be considering the **radial differential expressions** $\partial_r (-\Delta_r)^{m-1}$, and respectively $(-\Delta_r)^m$, for $m \in \mathbb{N}$, interpreting $\partial_r (-\Delta_r)^0 := \partial_r$.

4. Multi-dimensions and Distance to the Boundary

Given $m, n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, we also define the constants

$$A'(m,n,\alpha) := \prod_{j=1}^{\lfloor m/2 \rfloor} \left(\frac{n+2m-4j-\alpha}{2} \right)^2,$$

$$A''(m,n,\alpha) := \prod_{k=0}^{\lfloor (m-1)/2 \rfloor} \left(\frac{n-2m+4k+\alpha}{2} \right)^2,$$

$$A(m,n,\alpha) := A'(m,n,\alpha)A''(m,n,\alpha),$$

so that

$$A(m,n,\alpha) = \begin{cases} \prod_{j=1}^{(m-1)/2} \left(\frac{n-2+4j-\alpha}{2} \right)^2 \prod_{k=1}^{(m+1)/2} \left(\frac{n+2-4k+\alpha}{2} \right)^2, & m \text{ odd}, \\ \prod_{j=1}^{m/2} \left(\frac{n-4+4j-\alpha}{2} \right)^2 \prod_{k=1}^{m/2} \left(\frac{n-4k+\alpha}{2} \right)^2, & m \text{ even}. \end{cases}$$

and

$$A(m,1,\alpha) = A(m,\alpha).$$

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Induction over $m \in \mathbb{N}$, and **integration in polar coordinates**

$$\int_{\mathbb{R}^n} d^n x \, f(x) = \int_{S^{n-1}} d\omega(\theta) \int_0^\infty r^{n-1} dr \, f(r,\theta), \quad f \in L^1(\mathbb{R}^n), \ n \in \mathbb{N},$$

establishes the weighted Birman-type inequalities with radial refinements:

Theorem 4.1 (Weighted Birman w. Radial Refinements)

Fix $m, n \in \mathbb{N}$, and $\alpha \in \mathbb{R}$. For all $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$,

$$(i) \int_{\mathbb{R}^n} d^n x \, |x|^\alpha \big|(\partial_r(-\Delta_r)^{m-1} f)(x)\big|^2 \geq A(2m-1,n,\alpha) \int_{\mathbb{R}^n} d^n x \, |x|^{\alpha-4m+2} |f(x)|^2;$$

$$(ii) \int_{\mathbb{R}^n} d^n x \, |x|^\alpha \big|((-\Delta_r)^m f)(x)\big|^2 \geq A(2m,n,\alpha) \int_{\mathbb{R}^n} d^n x \, |x|^{\alpha-4m} |f(x)|^2.$$

Moreover, inequalities (i)–(ii) are strict for $f \not\equiv 0$.

Remark 4.2

As before, if $\alpha > 2m - n$, then $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$ in Theorem 4.1 can be replaced by $f \in C_0^\infty(\mathbb{R}^n)$.

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To conclude this talk, we briefly mention that, for the case $\alpha < 2 - n$, Theorem 4.1 generalizes further with the standard power weight $|x|^\alpha$, $\alpha \in \mathbb{R}$, replaced by the **shortest distance to the boundary $\delta(x)^\alpha$, $\alpha \in \mathbb{R}$**, defined below:

Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ be bounded and open with boundary $\partial\Omega = \overline{\Omega} \backslash \Omega^\circ$. The shortest distance function δ on Ω is given by

$$\delta(x) := \inf_{y \in \partial\Omega} |x - y|, \quad x \in \Omega.$$

Indeed, considering the special case $\Omega = B_n(0; \rho) \backslash \{0\}$ the punctured open ball in $\mathbb{R}^n$ of radius $\rho \in (0, \infty)$ centered at the origin, then using polar coordinates as before, and induction over $m \in \mathbb{N}$, the **distance-weighted Birman–Hardy–Rellich-type inequalities** hold on $C_0^\infty(B_n(0; \rho) \backslash \{0\})$.

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Theorem 4.3 (Radial Birman w. Shortest Distance to Boundary)

Fix $m, n \in \mathbb{N}$, $\rho \in (0, \infty)$, $\alpha \in \mathbb{R}$ with $\alpha < 2 - n$. The following hold:

(i) For all $f \in C_0^\infty(B_n(0;\rho)\backslash\{0\})$,

$$\int_{B_n(0;\rho)} d^n x \, \delta(x)^\alpha \big|(\partial_r(-\Delta_r)^{m-1} f)(x)\big|^2 \geq A(2m-1, n, \alpha) \int_{B_n(0;\rho)} d^n x \, \delta(x)^{\alpha-4m+2} |f(x)|^2.$$

(ii) For all $f \in C_0^\infty(B_n(0;\rho)\backslash\{0\})$,

$$\int_{B_n(0;\rho)} d^n x \, \delta(x)^\alpha \big|((-\Delta_r)^m f)(x)\big|^2 \geq A(2m, n, \alpha) \int_{B_n(0;\rho)} d^n x \, \delta(x)^{\alpha-4m} |f(x)|^2.$$

Moreover, inequalities (i)–(ii) are strict for $f \not\equiv 0$.

Thank You, Lance!!!