On another characterization of Askey-Wilson polynomials

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Introduction

The algebraic theory on OPS due to P. Maroni.

Let $\mathcal{P}$ be the space of all polynomials with complex coefficients. If $\mathcal{P}$ is equipped with his finer topology, then his algebraic dual $\mathcal{P}^*$, and his topological dual $\mathcal{P}'$ coincide: $\mathcal{P}^* = \mathcal{P}'$.

To any simple set of polynomials $\{P_n\}_{n\geq 0}$, we can associate a corresponding dual basis $\{a_n\}_{n\geq 0}$ in $\mathcal{P}'$ such that

$$\langle a_n, P_m \rangle = \delta_{n,m} \ .$$

In addition, every u in $\mathcal{P}'$ can be written as

$$u = \sum_{n=0}^{\infty} \langle u, P_n \rangle a_n$$

Orthogonal polynomial sequence (OPS)

Definition

Let $(P_n)_{n\geq 0}$ be a sequence in $\mathcal{P}$ and $u \in \mathcal{P}'$. Then

- $(P_n)_{n\geq 0}$ is an OPS with respect to u if $(P_n)_{n\geq 0}$ is a simple set and for all $n \in \mathbb{N}$, there is $h_n \in \mathbb{C} \setminus \{0\}$ such that

$$\langle u, P_n P_m \rangle = h_n \delta_{n,m}$$

.

Favard's Theorem

Monic orthogonal polynomials sequences (OPS) are characterized by a three term recurrence relation (TTRR) of the form

$$P_{n+1}(x) = (x - B_n)\, P_n(x) - C_n P_{n-1}(x)$$

with $P_{-1}(x) = 0$ where $C_n \neq 0$ for all $n \geq 1$.

Notations and definitions

A lattice is a mapping $x(s)$, $s \in \mathbb{C}$, given by (N. M. Atakishiev et al. 1995)

$$x(s) := \begin{cases} \mathfrak{c}_1 q^{-s} + \mathfrak{c}_2 q^{s} + \mathfrak{c}_3 & \text{if} \quad q \neq 1\,, \\ \mathfrak{c}_4 s^2 + \mathfrak{c}_5 s + \mathfrak{c}_6 & \text{if} \quad q = 1\,. \end{cases}$$

We define

$$\alpha_n := \frac{q^{n/2} + q^{-n/2}}{2}, \quad \gamma_n := \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}},$$

with $\alpha := \alpha_1$ and $\beta := (1-\alpha)\mathfrak{c}_3$ for $n = 0, 1, 2, \ldots$.

Notations and definitions

For $\mathbf{u} \in \mathcal{P}^*$, we define $D_x f$, $S_x f$, $\mathbf{D}_x\mathbf{u}$ and $\mathbf{S}_x\mathbf{u}$ by (A. F. Nikiforov, S. K. Suslov, V. B. Uvarov 1991 and M. Foupouagnigni 2011)

$$D_x f(x(s)) := \frac{f\left(x(s+\frac{1}{2})\right) - f\left(x(s-\frac{1}{2})\right)}{x(s+\frac{1}{2}) - x(s-\frac{1}{2})},$$

$$S_x f(x(s)) := \frac{f\left(x(s+\frac{1}{2})\right) + f\left(x(s-\frac{1}{2})\right)}{2},$$

$$\langle \mathbf{D}_x\mathbf{u},\ f\rangle = -\langle \mathbf{u},\ D_x f\rangle,\ \ \langle \mathbf{S}_x\mathbf{u},\ f\rangle = \langle \mathbf{u},\ S_x f\rangle,\ \ f \in \mathcal{P}.$$

Classical OPS on lattices

Definition

A regular functional $u \in \mathcal{P}'$ is called classical functional on lattices if it satisfies the following distributional equation

$$\mathbf{D}_x(\phi u) = \mathbf{S}_x(\psi u) \ ,$$

where ϕ and ψ are polynomials such that $deg(\phi) \leq 2$ and $deg(\psi) = 1$.

If $deg(\phi) \geq 0$ and $deg(\psi) \geq 1$, then u is called semi-classical functional.

Characterization of classical OPS on lattices

Let $u \in \mathcal{P}'$ be a regular functional and let $\{P_n\}_{n\geq 0}$ be its corresponding monic OPS. Then the following are equivalent:

C1 u is classical. That is : $\exists \phi \in \mathcal{P}_2$, $\exists \psi \in \mathcal{P}_1$ with $deg \psi = 1$ such that $\mathbf{D}_x(\phi u) = \mathbf{S}_x(\psi u)$

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- C2 $(D_x P_{n+1})_{n\geq 0}$ is an OPS,
- C3 $\forall n$, $\exists r_n$, $s_n \in \mathbb{C}$ such that

$$S_x P_n = a_n D_x P_{n+1} + b_n D_x P_n + c_n D_x P_{n-1} \quad (c_n \neq 0)$$

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- C4 (Bochner-type) $\exists \phi, \psi \in \mathcal{P}$, $\exists \lambda_n \in \mathbb{C}$ with $\lambda_n \neq 0$ if $n \geq 1$ such that

$$\phi D_x^2 P_n + \psi S_x D_x P_n + \lambda_n P_n = 0$$

[M. Foupouagnigni, M. Kenfack, S. Mboutngam, 2011, 2012], [R.S. Costas-Santos and F. Marcellán, 2009], ... etc.

Theorem 1: classical OPS on a q-quadratic lattice

Consider the lattice $z = \mathfrak{c}_1 q^{-s} + \mathfrak{c}_2 q^s + \mathfrak{c}_3$.

Regularity conditions

Let $\mathbf{u} \in \mathcal{P}^* \setminus \{\mathbf{0}\}$ be such that

$$\mathbf{D}_x\big((az^2+bz+c)\mathbf{u}\big) = \mathbf{S}_x\big((dz+e)\mathbf{u}\big).$$

Then $\mathbf{u}$ is regular if and only if

$$d_n \neq 0\,, \quad \phi^{[n]}\left(\mathfrak{c}_3 - \frac{e_n}{d_{2n}}\right) \neq 0\,, \quad \forall n \in \mathbb{N}_0,$$

where $d_n := a\gamma_n + d\alpha_n$, $e_n := (b+2a\mathfrak{c}_3)\gamma_n + (d\mathfrak{c}_3+e)\alpha_n$,

$$\phi^{[n]}(z) := \big(d(\alpha^2-1)\gamma_{2n}+a\alpha_{2n}\big)\big((z-\mathfrak{c}_3)^2-2\mathfrak{c}_1\mathfrak{c}_2\big)+\big((b+2a\mathfrak{c}_3)\alpha_n+ (d\mathfrak{c}_3+e)(\alpha^2-1)\gamma_n\big)(z-\mathfrak{c}_3)+a\mathfrak{c}_3^2+b\mathfrak{c}_3+2a\mathfrak{c}_1\mathfrak{c}_2+c.$$

Generalization for the lattice $z = \mathfrak{c}_1 q^{-s} + \mathfrak{c}_2 q^s + \mathfrak{c}_3$

Under these regularity conditions, the recurrence coefficients for the monic OPS $(P_n^{[k]})_{n\geq 0}$ are given by

$$B_n^{[k]} = \mathfrak{c}_3 + \frac{\gamma_n e_{n+k-1}}{d_{2n+2k-2}} - \frac{\gamma_{n+1} e_{n+k}}{d_{2n+2k}} ,$$

$$C_{n+1}^{[k]} = -\frac{\gamma_{n+1} d_{n+2k-1}}{d_{2n+2k-1} d_{2n+2k+1}} \phi^{[n+k]} \left(\mathfrak{c}_3 - \frac{e_{n+k}}{d_{2n+2k}} \right) \quad (n, k \in \mathbb{N}_0) .$$

Here $P_n^{[k]} = \frac{\gamma_n!}{\gamma_{n+k}!} D_x^k P_{n+k}$, and

$$P_{-1}^{[k]}(z) = 0 , \quad zP_n^{[k]} = P_{n+1}^{[k]} + B_n^{[k]} P_n^{[k]} + C_n^{[k]} P_{n-1}^{[k]} .$$

Theorem 2: classical OPS on a quadratic lattice

Let $x(s)$ be the quadratic lattice $z = x(s) = 4\beta s^2 + \mathfrak{c}_5 s + \mathfrak{c}_6$.

Regularity conditions

Let $\mathbf{u} \in \mathcal{P}^* \setminus \{\mathbf{0}\}$ be such that

$$\mathbf{D}_x\big((az^2+bz+c)\mathbf{u}\big) = \mathbf{S}_x\big((dz+e)\mathbf{u}\big).$$

Then **u** is regular if and only if

$$d_n \neq 0\,, \quad \phi^{[n]}\left(-\beta n^2 - \frac{e_n}{d_{2n}}\right) \neq 0\,, \quad \forall n \in \mathbb{N}_0\,.$$

where where $d_n := an + d$, $e_n := bn + e + 2d\beta n^2$, $\psi(z) = dz + e$ and

$$\phi^{[n]}(z) := az^2 + (b + 6\beta n d_n)z + a\beta^2 n^4 + b\beta n^2 + c + 2\beta n\psi(\beta n^2) - n\left(4\beta\mathfrak{c}_6 - \tfrac{1}{4}\mathfrak{c}_5^2\right) d_n.$$

Generalization for the lattice $x(s) = 4\beta s^2 + \mathfrak{c}_5 s + \mathfrak{c}_6$

Under these regularity conditions, the recurrence coefficients of the monic OPS $(P_n^{[k]})_{n\geq 0}$ are given by

$$B_n^{[k]} = \frac{n e_{n+k-1}}{d_{2n+2k-2}} - \frac{(n+1)e_{n+k}}{d_{2n+2k}} - 2\beta\left((n+k)^2 - n - \frac{1}{2}k^2\right) ,$$

$$C_{n+1}^{[k]} = -\frac{(n+1)d_{n+2k-1}}{d_{2n+2k-1}d_{2n+2k+1}}\phi^{[n+k]}\left(-\beta(n+k)^2 - \frac{e_{n+k}}{d_{2n+2k}}\right) .$$

Application 1: for the lattice $x(s) = \mathfrak{c}_1 q^{-s} + \mathfrak{c}_2 q^s + \mathfrak{c}_3$

Consider the following equation.

$$\pi(z) \mathcal{D}_x P_n(z) = (a_n z + b_n) P_n(z) + c_n P_{n-1}(z)\,,\ c_n \neq 0. \tag{1}$$

Conjecture 24.7.8, M. E. H. Ismail (2005)

Up to an affine transformation of the variable, the only monic OPS $(P_n)_{n\geq 0}$ satisfying (1) are the continuous $q-$Jacobi polynomials or the Al-Salam-Chihara polynomials or special or limiting cases of them.

Sketch of the Proof

We assume that $0 < q < 1$. Let $\mathbf{u} \in \mathcal{P}^*$ be the regular functional with respect to the monic OPS $(P_n)_{n\geq 0}$ satisfying (1).

I. **u** is x-classical. Moreover $D_x(\phi\mathbf{u}) = S_x(\psi\mathbf{u})$, where ϕ and ψ are polynomials given by

$$\phi(z) := (\mathfrak{a}z - \mathfrak{b})(z - B_0) - (\mathfrak{a} + \alpha)C_1, \quad \psi(z) := z - B_0,$$

with

$$\mathfrak{a} := \frac{(a_2 C_2 + c_2)C_1}{(a_1 C_1 + c_1)C_2} - \alpha, \quad \mathfrak{b} := \beta - B_0 + (\mathfrak{a} + \alpha)B_1 - \frac{b_1 + a_1 B_1}{c_1 + a_1 C_1} C_1 .$$

Sketch of the proof

II. The coefficients B_n and C_n of the TTRR satisfied by $(P_n)_{n\geq 0}$ fulfill the following system of equations:

$$a_{n+2} - 2\alpha a_{n+1} + a_n = 0 \ ,$$

$$t_{n+2} - 2\alpha t_{n+1} + t_n = 0 \ , \ t_n := c_n/C_n, \ r_n := t_n + a_n - a_{n-1},$$

$$r_{n+3}(B_{n+2} - \mathfrak{c}_3) - (r_{n+2} + r_{n+1})(B_{n+1} - \mathfrak{c}_3) + r_n(B_n - \mathfrak{c}_3) = 0,$$

$$(r_{n+1} + r_{n+2})(C_{n+1} - \mathfrak{c}_1\mathfrak{c}_2) - 2(1+\alpha)r_n(C_n - \mathfrak{c}_1\mathfrak{c}_2) + (r_{n-1} + r_{n-2})\times$$
$$(C_{n-1} - \mathfrak{c}_1\mathfrak{c}_2) = r_n\Big[(B_n - \mathfrak{c}_3)^2 - 2\alpha(B_n - \mathfrak{c}_3)(B_{n-1} - \mathfrak{c}_3) + (B_{n-1} - \mathfrak{c}_3)^2\Big],$$

Sketch of the proof

$$\Big(2(1-\alpha)(a_nB_n+b_n)-4\beta a_n\Big)B_n^2+(t_{n+1}+a_{n+1}-a_{n+2})B_{n+1}C_{n+1}$$
$$+(t_n+a_{n-1}-a_{n-2})B_{n-1}C_n+[(2a_n-a_{n+2}-a_{n-1})C_{n+1}$$
$$+(2a_n-a_{n+1}-a_{n-2})C_n+(1-2\alpha)(c_n+c_{n+1})-4\beta b_n+(\beta^2-\delta)a_n]\,B_n$$
$$+2\Big(b_n-\alpha b_{n+1}-\beta(a_n+a_{n+1}+t_{n+1})\Big)C_{n+1}$$
$$+2\Big(b_n-\alpha b_{n-1}-\beta(a_{n-1}+a_n+t_n)\Big)C_n=b_n(\delta-\beta^2),$$

where $\delta=\mathtt{U}_2(2\mathfrak{c}_3)$.

Sketch of the proof: case $\deg \pi = 0$

III. Solve the system separately according to the degree of $\deg \pi$. Recall the monic Al-Salam Chihara polynomials $(Q_n(z;c,d|q))_{n\geq 0}$:

$$B_n := \frac{1}{2}(c+d)q^n, \quad C_{n+1} := \frac{1}{4}(1-cdq^n)(1-q^{n+1}) \, .$$

Up to an affine transformation of the variable, the only monic OPS $(P_n)_{n\geq 0}$ satisfying $D_x P_n(z) = c_n P_{n-1}(z)$, $n \in \mathbb{N}_0$, are the monic Al-Salam Chihara polynomials with parameters $c = d = 0$:

$$P_n(z) = 2^n(\mathfrak{c}_1\mathfrak{c}_2)^{n/2} Q_n\left(\frac{z-\mathfrak{c}_3}{2\sqrt{\mathfrak{c}_1\mathfrak{c}_2}}; 0, 0 \middle| q\right), \quad n \in \mathbb{N}_0.$$

Sketch of the proof: case deg $\pi = 1$

Let r be a complex number.

Up to an affine transformation of the variable, the only OPS $(P_n)_{n\geq 0}$ satisfying $(z-\mathfrak{c}_3-r)D_xP_n(z)=b_nP_n(z)+c_nP_{n-1}(z)$, $n\in\mathbb{N}_0$, are the monic Al-Salam Chihara polynomials with parameters c and d both nonzero and satisfying $c/d=q^{\pm 1/2}$:

$$P_n(z)=2^n(\mathfrak{c}_1\mathfrak{c}_2)^{n/2}Q_n\left(\frac{z-\mathfrak{c}_3}{2\sqrt{\mathfrak{c}_1\mathfrak{c}_2}};c,d\Big|q\right),\quad n\in\mathbb{N}_0\ .$$

Remark: The monic Continuous q-Laguerre polynomial is special case of the solution.

Sketch of the proof: case $\deg \pi = 2$

Let's consider $r, s \in \mathbb{C}$. Recall the monic continuous q-Jacobi polynomials $(P_n^{(a,b)}(z|q))_{n\geq 0}$:

$$B_n := \frac{(1+q^{1/2})(1-q^{(a+b)/2})(q^{a/2}-q^{b/2})q^{n+1/4}}{2(1-q^{n+(a+b)/2})(1-q^{n+1+(a+b)/2})},$$

$$C_n := \frac{(1-q^n)(1-q^{n+a+b})(1-q^{n+a})(1-q^{n+b})}{4(1-q^{n+(a+b-1)/2})(1-q^{n+(a+b)/2})^2(1-q^{n+(a+b+1)/2})}.$$

Up to an affine transformation of the variable, the only OPS satisfying $(z-\mathfrak{c}_3-r)(z-\mathfrak{c}_3-s)D_x P_n(z) = (a_n z + b_n)P_n(z) + c_n P_{n-1}(z)$, for each $n \in \mathbb{N}_0$, are the Chebyshev polynomials of the first kind and the continuous q-Jacobi polynomials:

$$P_n(z) = 2^n(\mathfrak{c}_1\mathfrak{c}_2)^{n/2} P_n^{(a,b)}\left(\frac{z-\mathfrak{c}_3}{2\sqrt{\mathfrak{c}_1\mathfrak{c}_2}}\Big| q\right), \quad n \in \mathbb{N}_0.$$

Other solved conjecture

Classical orthogonal polynomials on lattices are characterized by the following equation [M. Kenfack, K. Jordaan, 2018, 2019]

$$\pi(z)\mathrm{D}_x^2 P_n(z) = \sum_{j=-2}^{2} a_{n,j} P_{n+j}(z) \qquad (n = 0, 1, \ldots) \ .$$

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$(P_n)_{n\geq 0}$ and $(\mathrm{D}_x^k P_{n+k})_{n\geq 0}$ are OPS for some fixed k. Then P_n is Askey-Wilson or special or limiting case.

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$(P_n)_{n\geq 0}$ and $(\mathrm{D}_x^k P_{n+k})_{n\geq 0}$ are OPS for some fixed k. Then P_n is Askey-Wilson or special or limiting case.

More generally we have

$$\pi(z)\mathrm{D}_x^k P_n(z) = \sum_{j=-m}^{m} a_{n,j} P_{n+j}(z) \qquad (n = 0, 1, \ldots) ,$$

with $\deg \pi \leq 2k$.

Application 2: For the lattice $x(s) = \mathfrak{c}_1 q^{-s} + \mathfrak{c}_2 q^s + \mathfrak{c}_3$

Theorem 3

If $(P_n)_{n\geq 0}$ is a monic OPS such that

$$\pi(z) D_x P_n(z) = a_n S_x P_{n+1}(z) + b_n S_x P_n(z) + c_n S_x P_{n-1}(z) \ ,$$

where π is a well chosen polynomial of degree at most two and $c_n \neq 0$, then $(P_n)_{n\geq 0}$ are multiple of Askey-Wilson polynomials, or special or limiting cases of them.

The monic Askey-Wilson polynomial

The coefficients of the TTRR satisfied by the monic Askey-Wilson polynomials $(Q_n(z,a,b,c,d|q))_{n\geq 0}$ are given by

$$B_n = \frac{1}{2}\left[a+\frac{1}{a}-\frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})} - \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-1})(1-abcdq^{2n-2})}\right],$$

$$C_{n+1} = (1-q^{n+1})(1-abcdq^{n-1})(1-abq^n) \times \frac{(1-acq^n)(1-adq^n)(1-bcq^n)(1-bdq^n)(1-cdq^n)}{4(1-abcdq^{2n-1})(1-abcdq^{2n})^2(1-abcdq^{2n+1})}.$$

Case where $\deg \pi = 0$

Theorem 3-a

Up to an affine transformation of the variable, the only monic OPS, $(P_n)_{n\geq 0}$, satisfying

$$D_x P_{n+1}(z) = \alpha_n^{-1}\gamma_{n+1} S_x P_n(z),$$

are those of the Askey-Wilson polynomials

$$P_n(z) = 2^n(\mathfrak{c}_1\mathfrak{c}_2)^{n/2} Q_n\left(\frac{z-\mathfrak{c}_3}{2\sqrt{\mathfrak{c}_1\mathfrak{c}_2}}; a, -a, iq^{-1/2}/a, -iq^{-1/2}/a \Big| q\right),$$

with $a \notin \{\pm q^{(n-1)/2}, \pm i q^{-n/2} \,|\, n = 0, 1, \ldots\}$.

Sketch of the proof

We first prove that

$$\mathbf{D}_x\Big((\gamma_{n+1}\mathbb{U}_1 P_{n+1} + \alpha_n C_{n+1} P_n)\mathbf{u}\Big) = -\alpha\gamma_{n+1}\mathbf{S}_x\big(P_{n+1}\mathbf{u}\big). \quad (2)$$

Indeed let $(\mathbf{a}_n)_{n\geq 0}$ and $(\mathbf{a}_n^{[1]})_{n\geq 0}$ be the dual basis associated to the sequences $(P_n)_{n\geq 0}$ and $(P_n^{[1]})_{n\geq 0}$, respectively. Let

$$zP_n = P_{n+1} + B_n P_n + C_n P_{n-1}$$

be the TTRR satisfied by $(P_n)_{n\geq 0}$. Then we have

$$D_x P_{n+1} = r_n S_x P_n \qquad \Rightarrow \qquad \mathbf{S}_x \mathbf{a}_n^{[1]} = \alpha_n \mathbf{a}_n.$$

We then apply D_x to above equation to obtain (2).

Sketch of the proof

We denote by $\mathbf{u}$ the regular functional whose $(P_n)_{n\geq 0}$ is the corresponding monic OPS. Taking $n = 0$ in (2), we see that $\mathbf{u}$ is x-classical. Moreover,

$$\mathbf{D}_x(\phi\mathbf{u}) = \mathbf{S}_x(\psi\mathbf{u}),$$

where

$$\psi(z) = z - B_0, \quad \phi(z) = -(\alpha - \alpha^{-1})(z - \mathfrak{c}_3)(z - B_0) - \alpha^{-1}C_1.$$

We now use the initials conditions in the given equation to find a possible relation between B_0 and C_1.

Sketch of the proof

From the given equation we obtain

$$B_n = \mathfrak{c}_3 + \frac{\alpha}{\alpha_{n-1}\alpha_n}(B_0 - \mathfrak{c}_3).$$

But from the theory developed, we have

$$B_n = \mathfrak{c}_3 + (1+q)(B_0 - \mathfrak{c}_3)q^{n-2}\frac{(q-1)(1-q^{2n-2}) + (1+q)q^{n-1}}{(1+q^{2n-3})(1+q^{2n-1})}.$$

Assume that $0 < q < 1$, then we obtain

$$2(1+q^{-1})(B_0 - \mathfrak{c}_3) = \lim_{n\to+\infty} q^{-n}(B_n - \mathfrak{c}_3) = (1-q^{-2})(B_0 - \mathfrak{c}_3).$$

Therefore

$$B_0 = \mathfrak{c}_3.$$

Sketch of the proof

We now apply the general formula to obtain

$$
\begin{aligned}
B_n &= \mathfrak{c}_3 \\
C_{n+1} &= \mathfrak{c}_1\mathfrak{c}_2 \frac{(1+q^{n-2})(1-q^{n+1})(1+rq^n)(1-r^{-1}q^{n-1})}{(1+q^{2n-2})(1+q^{2n})}
\end{aligned}
$$

where

$$
C_1 = \frac{1}{2}(1-q^{-1})(1+r^{-1})(1-rq)\mathfrak{c}_1\mathfrak{c}_2 \,,
$$

and the result follows.

Case where $\deg \pi = 1$

Theorem 3-b

Let $(P_n)_{n\geq 0}$ be a monic OPS satisfying

$$(z - \mathfrak{c}_3 - r)D_x P_n(z) = b_n S_x P_n(z) + c_n S_x P_{n-1}(z),$$

with $c_n \neq 0$ for $n = 0, 1, \ldots$, where the constant parameter r is chosen such that

$$2(2\alpha^2 - 1)\left(C_2 + \alpha(B_1 - B_0)r\right) = (B_1 - \mathfrak{c}_3)^2 - (B_0 - \mathfrak{c}_3)^2 \ .$$

Then P_n is a specific case of the monic Askey-Wilson polynomial:

$$P_n(z) = 2^n(\mathfrak{c}_1\mathfrak{c}_2)^{n/2} Q_n\left(\frac{z - \mathfrak{c}_3}{2\sqrt{\mathfrak{c}_1\mathfrak{c}_2}}; a_1, a_2, a_3, a_4 \middle| q\right) \ ,$$

Case where $\deg \pi = 1$

where a_1, a_2, a_3 and a_4 are complex numbers solutions of the following equation

$$Z^4 - RZ^3 + TZ^2 - SZ - q^{-1} = 0\,,$$

with

$$(R,T,S) \in \left\{ \left(2\frac{qB_0 - B_1}{q-1}, T_1, 2\frac{q^{-1}B_0 - B_1}{q-1}\right), \left(2\frac{(q+1)B_0}{q-1}, \frac{4(3\alpha^2-1)}{q-1}, 2\frac{(1+q^{-1})B_0}{q-1}\right) \right\}\,,$$

where

$$T_1 = 1 - q^{-1} + 8\frac{(B_0^2-\alpha^2)((4\alpha^2-3)B_1 - B_0)}{(B_1+(4\alpha^2-1)B_0)(q-1)} - 4\frac{(B_1-B_0)B_0}{q-1}\,.$$

Some References

- D. Mbouna and A. Suzuki, On another characterization of Askey-Wilson polynomials, arXiv:2202.10167 [math.CA] (2022).
- K. Castillo, D. Mbouna, and J. Petronilho: *On the functional equation for classical orthogonal polynomials on lattices*, arXiv:2102.00033 [math.CA] (2021).
- K. Castillo, D. Mbouna, and J. Petronilho, A characterization of continuous q-Jacobi, Chebyshev of the first kind and Al-Salam Chihara polynomials, arXiv:2109.06147 [math.CA] (2021).
- N. M. Atakishiev, M. Rahman, and S. K. Suslov, On classical orthogonal polynomials, Constr. Approx. **11** (1995) 181–226.

- M. E. H. Ismail: Classical and quantum orthogonal polynomials in one variable. With two chapters by W. Van Assche. With a foreword by R. Askey., Encyclopedia of Mathematics and its Applications **98**. Cambridge University Press, Cambridge (2005).
- P. Maroni: Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques, In Orthogonal Polynomials and Their Applications, C. Brezinski, L. Gori, and A. Ronveaux, Eds.,
- M. Foupouagnigni, M. Kenfack-Nangho, and S. Mboutngam, *Characterization theorem of classical orthogonal polynomials on non-uniform lattices: the functional approach*, Integral Transforms Spec. Funct. **22** (2011) 739-758.

THANK YOU !!