Localization and Eigenvalue Statistics within Hartree-Fock theory

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Partially Supported by NSF DMS-2000345 and DMS-2052572.

Baylor Analysis Fest, May 23 2022

Background: The Anderson model

The Anderson model on $\ell^2\left(\mathbb{Z}^d\right)$

$$H_\omega = -\Delta + \lambda V_\omega$$

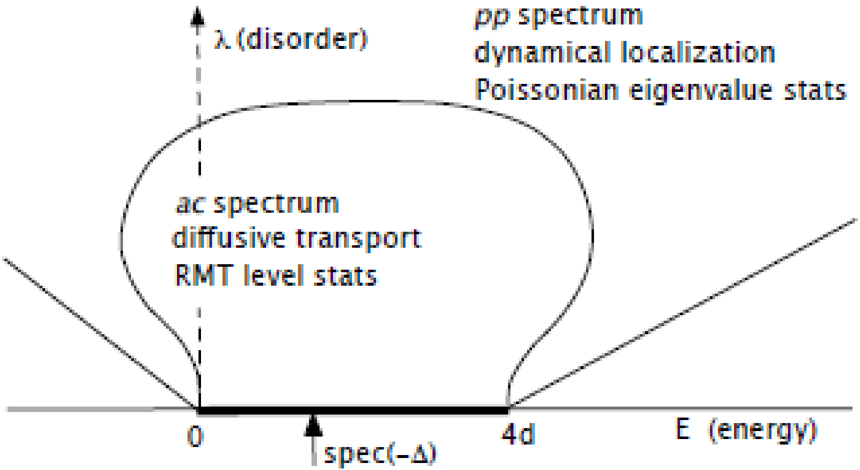
- $\left(\Delta\varphi\right)(n) = \sum_{|m-n|_1=1}\left(\varphi(m) - \varphi(n)\right)$, $\;|n|_1 = |n_1| + \cdots + |n_d|$.
- $\left(V_\omega\varphi\right)(n) = \omega(n)\varphi(n)$
- $\{\omega(n)\}_{n\in\mathbb{Z}^d}$ independent, identically distributed random variables.
- $\lambda > 0$ denotes the disorder strength.

Background: The Anderson model

$$H_\omega = -\Delta + \lambda V_\omega$$

- Philip Anderson (1958): Disorder may drastically affect the transport properties of an environment.
- Anderson localization: suppression of electron transport due to disorder.
- Dynamical localization: typical decay of matrix elements of e^{-itH_ω}

Predicted Phase Diagram



Picture source: Aizenman-Warzel book "Random Operators: disorder effects on quantum spectra and dynamics."

Background: The Anderson model

- For $H_\omega = -\Delta + \lambda V_\omega$ and $d \geq 2$ localization is well understood at (i) large disorder ($\lambda >> 1$) for all energies and (ii) at any $\lambda > 0$ near band edges.
- When $d = 1$ special tools are available (Furstenberg, Ishii-Pastur and Kotani-Simon theorems) and dynamical localization holds for *any* $\lambda > 0$. Contributions by Goldsheid-Molchanov-Pastur, Kunz-Souillard, Carmona-Klein-Martinelli, Aizenman-Warzel, Bucaj-Damanik-Fillman-Gerbuz-VandenBoom-Wang-Zhang, Jitormirskaya-Zhu, ...
- For $d \geq 2$ the techniques are either based on the multiscale analysis, initiated by Fröhlich and Spencer (1983) and developed further by Klein and co-authors, or the Aizenman-Molchanov fractional moment method (1993).

Random operators within Hartree-Fock theory

$$(H_\omega \varphi)(n) = -\sum_{m\sim n} (\varphi(m) - \varphi(n)) + \lambda\omega(n)\varphi(n) + gV_{\text{eff}}(n)\varphi(n)$$

where V_{eff} is the effective potential defined implicitly by

$$V_{\text{eff}}(n) = \langle \delta_n, F(H_\omega)\delta_n \rangle, \quad F(z) = \frac{1}{1+e^{\beta z}}.$$

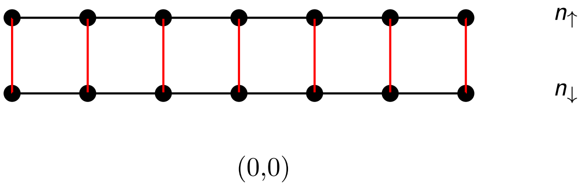
- When $g \neq 0$, $H_\omega = -\Delta + \lambda V_\omega + gV_{\text{eff}}$ takes into account interactions among particles.
- For $|g| << 1$, V_{eff} exists and is unique by a fixed point argument. It follows that H_ω is an ergodic Schrödinger operator. It is also random and implicitly defined. It is a nonlinear and non-local function of H_ω.

The Hubbard Model

$$\Lambda \subset \mathbb{Z}^d$$

$$\Lambda \times \{-1, 1\}$$

$$n_\uparrow := (n, 1) \quad n_\downarrow := (n, -1)$$



$$H_{\mathrm{Hub}}(\omega) = \begin{pmatrix} H_\uparrow(\omega) \\ H_\downarrow(\omega) \end{pmatrix} := \begin{pmatrix} -\Delta + \lambda\omega(n) + gV_\uparrow(n) & 0 \\ 0 & -\Delta + \lambda\omega(n) + gV_\downarrow(n) \end{pmatrix}$$

Acting on $\ell^2\left(\mathbb{Z}^d\right) \oplus \ell^2\left(\mathbb{Z}^d\right)$.

1. $$\begin{pmatrix} V_\uparrow(\omega)(n) \\ V_\downarrow(\omega)(n) \end{pmatrix} = \begin{pmatrix} \langle \delta_n, F(H_\downarrow)\delta_n \rangle \\ \langle \delta_n, F(H_\uparrow)\delta_n \rangle \end{pmatrix}.$$
2. Motivation comes from the formalism developed by Bach-Lieb-Solovej (93) on the Hubbard model within *generalized* Hartree-Fock theory.
3. For simplicity, from now on we look at $H_\omega = -\Delta + \lambda V_\omega + gV_{\mathrm{eff}}$.

Theorem One, Localization at Large Disorder and Weak Interactions

Under additional assumptions on the probability distribution of the random potential, we have

Theorem (M. and Schenker, 2019)

Whenever $|g| << 1$

$$\mathbb{E}\left(\sup_t |\langle \delta_n, e^{-itH_\omega}\delta_0\rangle|\right) \leq Ce^{-\nu|n|}$$

holds for some $C, \nu > 0$ *in the following regimes:*

- A. *if* $d \geq 2$, *whenever* $\lambda >> 1$.
- B. *if d=1, for any* $\lambda > 0$.

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Dynamics can be replaced by eigenfunction correlators

$$Q(m,n) := \sup_{|\varphi|\leq 1} |\langle \delta_m, \varphi(H)\delta_n\rangle|.$$

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holds in the following regimes (recall that d *denotes the lattice dimension):*

A. *if* $d \geq 2$, *whenever* $\lambda >> 1$.
B. *if d=1, for any* $\lambda > 0$.

In particular, for almost every ω, H_ω has pure point spectrum with exponentially decaying eigenfunctions.

Poisson Statistics

- Let $\Lambda_L = [-L, L]^d \cap \mathbb{Z}^d$ and $H_L = \mathbb{1}_L H_\omega \mathbb{1}_L$ be the restriction of $H_\omega = -\Delta + \lambda V_\omega + g V_{\text{eff}}$ to $\ell^2(\Lambda_L)$.
- Given $E \in \mathbb{R}$ we study the point process $\{|\Lambda_L|(E_{n,L} - E) : n \in \mathbb{N}\}$ of rescaled eigenvalues.

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- Given $E \in \mathbb{R}$ we study the point process $\{|\Lambda_L|(E_{n,L} - E) : n \in \mathbb{N}\}$ of rescaled eigenvalues.
- Let $I \subset \mathbb{R}$ be an and $\mu_L^E(I;\omega)$ be the number of eigenvalues of H_L in the interval $E + \frac{1}{|\Lambda_L|} I$.

$$\mu_L^E(I;\omega) = \sum_n \delta_{|\Lambda_L|(E_{n,L}(\omega)-E)}(I)$$

$\mu_L^E(\cdot;\omega)$ is the random counting measure.

Poisson Statistics

Theorem (M. In preparation)

For energies $E \in \mathbb{R}$ *in the exponential localization regime of* $H_\omega = -\Delta + \lambda V_\omega + g V_{\text{eff}}$, μ_L^E *converges in distribution, as* $L \to \infty$, *to a Poisson process with density given by the density of states* $\nu(E)$.

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Thus

- $\lim_{L\to\infty} \mathbb{E}\left(e^{-\mu_L^E(\varphi)}\right) = \mathbb{E}\left(e^{-\mu(\varphi)}\right)$ for all $\varphi \in C_c^+(\mathbb{R})$
- $\mathbb{P}\left(\mu(I) = k\right) = \frac{\bar{\mu}(I)^k}{k!} e^{-\bar{\mu}(I)}$ holds for each Borel set I where $\bar{\mu}(I) = \nu(E)|I|$ and $\nu(E) = \frac{\mathbb{E}(\langle \delta_{\mathbf{0}}, P_{dE}(H)\delta_{\mathbf{0}}\rangle)}{dE}$.

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In particular, this extends the result of Minami to the interacting context and shows that localization portion of the *Spectral Statistics Conjecture* persists under weak interactions!

Thank you!

Pressure Functional

Let $\Lambda \subset \mathbb{Z}^d$ be finite and consider the pressure functional

$$-\mathcal{P}(\Gamma) = \mathcal{E}(\Gamma) - \beta^{-1}\mathcal{S}(\Gamma).$$

acting on all matrices Γ of the form

$$\Gamma = \begin{pmatrix} \Gamma_\uparrow & 0 \\ 0 & \Gamma_\downarrow \end{pmatrix}$$

satisfying $0 \leq \Gamma \leq 1$, with $\Gamma_\uparrow$ and $\Gamma_\downarrow$ acting on $\ell^2(\Lambda)$.

Pressure Functional

$$-\mathcal{P}(\Gamma) = \mathcal{E}(\Gamma) - \beta^{-1}\mathcal{S}(\Gamma).$$

The Energy functional is

$$\mathcal{E}(\Gamma) = \mathrm{Tr}\left(-\Delta - \mu + \lambda V_{\omega}\right)\Gamma + g\sum_{n}\langle \delta_n, \Gamma_{\uparrow}\delta_n\rangle\langle \delta_n, \Gamma_{\downarrow}\delta_n\rangle.$$

The Entropy is given by

$$\mathcal{S}(\Gamma) = -\mathrm{Tr}\left(\Gamma\log\Gamma + (1-\Gamma)\log(1-\Gamma)\right).$$

Motivation

Minimizer $\Gamma = \begin{pmatrix} \Gamma_\uparrow & 0 \\ 0 & \Gamma_\downarrow \end{pmatrix}$ with $0 \le \Gamma \le 1$, with $\Gamma_\uparrow$ and $\Gamma_\downarrow$ acting on $\ell^2(\Lambda)$. of $-\mathcal{P}(\Gamma)$ satisfies

$$\langle \delta_n, \Gamma_\uparrow \delta_n \rangle = \langle \delta_n, \frac{1}{1 + e^{\beta(-\Delta - \mu + \lambda V_\omega + \Gamma_\downarrow)}} \delta_n \rangle$$

$$\langle \delta_n, \Gamma_\downarrow \delta_n \rangle = \langle \delta_n, \frac{1}{1 + e^{\beta(-\Delta - \mu + \lambda V_\omega + \Gamma_\uparrow)}} \delta_n \rangle.$$