uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

Coherent pairs of measures on the unit circle.

F. Marcellán

The Baylor Analysis Fest, Waco, TX, May 23-27, 2022

Contents

1. OPRL
2. Semiclassical OPRL
3. Coherent pairs of measures and Sobolev OP
4. OPUC
5. Coherent pairs of measures on the unit circle
6. Coherent pairs of measures of the second kind
7. Sobolev OPUC

OPRL

Let u be a linear functional in the linear space of polynomials in one variable with real coefficients.

$$u_n = \langle u, x^n \rangle, n \geq 0,$$

is the moment of n-th order associated with u.
An example of linear functional is associated with a positive Borel measure μ supported on a set E of the real line.

$$\langle u, p(x) \rangle = \int_E p(x) d\mu(x). \tag{1}$$

A bilinear form can be defined from u as follows.

$$(p(x), q(x)) = \langle u, p(x)q(x) \rangle \tag{2}$$

OPRL

For the linear functional u you can define the Gram matrix H associated with (,). Taking into account $(xp(x), q(x)) = (p(x), xq(x))$, H is a Hankel matrix.

Definition

The linear functional u is said to be quasi-definite if the leading principal submatrices H_n of H are nonsingular for every $n \geq 0$. If H_n is positive-definite for every $n \geq 0$, then the linear functional is said to be positive-definite.

Theorem

If u is a quasi-definite linear functional, there exists a unique sequence of monic polynomials $\{P_n(x)\}_{n\geq 0}$ such that $deg(P_n) = n$ and $(P_n(x), P_m(x)) = k_n\delta_{m,n}, k_n \neq 0$.

These polynomials satisfy a three term recurrence relation (TTRR)

$$xP_n(x) = P_{n+1}(x) + b_nP_n(x) + c_nP_{n-1}(x), n \geq 0, P_{-1}(x) = 0, P_0(x) = 1.$$

where $b_n = \frac{\langle u, xP_n^2(x)\rangle}{\langle u, P_n^2(x)\rangle}, n \geq 0, \quad c_n = \frac{\langle u, P_n^2(x)\rangle}{\langle u, P_{n-1}^2(x)\rangle}, n \geq 1,$

Semiclassical linear functionals

Definition

A quasi-definite linear functional u is said to be semiclassical if there exist polynomials $\Phi(x), \Psi(x)$, $r = deg(\Phi), t = deg(\Psi) \geq 1$, such that $(\Phi(x)u)' = \Psi(x)u$ (in a distributional sense).

Given a semiclassical linear functional u , it satisfies an infinity number of distributional equations taking into account that for every polynomial $\chi(x)$ $(\chi\Phi u)' = (\chi\Psi - \chi'\Phi)u$.

Definition

The class s of a semiclassical linear functional u is defined as the minimum of the set of non-negative integers $h(u) = \{max(deg(\Phi) - 2, deg(\Psi) - 1), (\Phi(x)u)' = \Psi(x)u\}$.

Notice that the pair of polynomials realizing the class is unique up to a constant factor.

Coherent pairs of measures and Sobolev OP

The coherent pairs of measures were introduced in 1991 by A. Iserles, P. E. Koch, S. Nørsett and J. Sanz-Serna in the framework of the Sobolev the inner product

$$\langle f, g\rangle_\lambda = \int_a^b f(x)g(x)d\mu_0(x) + \lambda \int_a^b f'(x)g'(x)d\mu_1(x), \tag{3}$$

where $-\infty \leq a < b \leq \infty$, μ_0 and μ_1 are positive Borel measures on the real line with finite moments of all orders. Let $P_n(x; d\mu_i)$ denote the monic orthogonal polynomial of degree n with respect to $d\mu_i$, $i = 0, 1$.

Definition

The pair $\{d\mu_0, d\mu_1\}$ is called coherent if there exists a sequence of nonzero real numbers $\{a_n\}_{n\geq 1}$ such that

$$P_n(x; d\mu_1) = \frac{P'_{n+1}(x; d\mu_0)}{n+1} + a_n \frac{P'_n(x; d\mu_0)}{n}, \qquad n \geq 1. \tag{4}$$

Coherent pairs of measures and Sobolev OP

Definition (cont.)

If $[a,b]=[-c,c]$ and $d\mu_0$ and $d\mu_1$ are both even, then $\{d\mu_0, d\mu_1\}$ is called a symmetrically coherent pair if

$$P_n(x;d\mu_1)=\frac{P'_{n+1}(x;d\mu_0)}{n+1}+a_n\frac{P'_{n-1}(x;d\mu_0)}{n-1},\quad n\geq 2. \tag{5}$$

In the case of $d\mu_1=d\mu_0$, we call $d\mu_0$ self-coherent.

Theorem

If $\{d\mu_0, d\mu_1\}$ is a coherent pair, then

$$S_n(x;\lambda)+b_{n-1}(\lambda)S_{n-1}(x;\lambda)=P_n(x;d\mu_0)+\widehat{a}_{n-1}P_{n-1}(x;d\mu_0), \tag{6}$$

where $\widehat{a}_{n-1}=na_n/(n-1)$ *and* $b_{n-1}(\lambda)=\widehat{a}_{n-1}\|P_{n-1}(\cdot;d\mu_0)\|^2_{d\mu_0}/\|S_{n-1}(\cdot;\lambda)\|^2_{\lambda}$.

Coherent pairs of measures and Sobolev OP

Theorem

If $\{\mathcal{U}_0, \mathcal{U}_1\}$ is a coherent pair, then at least one of them has to be classical in the extended sense.

When $\mathcal{U}_0$ and $\mathcal{U}_1$ are positive definite linear functionals associated with measures $d\mu_0$ and $d\mu_1$, these cases are given as follows (Meijer 1997):

Laguerre case

1. $d\mu_0(x) = (x-\xi)x^{\alpha-1}e^{-x}dx$ and $d\mu_1(x) = x^{\alpha}e^{-x}dx$, where if $\xi < 0$ then $\alpha > 0$, and if $\xi = 0$ then $\alpha > -1$.
2. $d\mu_0(x) = x^{\alpha}e^{-x}dx$ and $d\mu_1(x) = \frac{x^{\alpha+1}e^{-x}}{x-\xi}dx + M\delta_{\xi}$, where if $\xi < 0$, $\alpha > -1$ and $M \geq 0$.
3. $d\mu_0(x) = e^{-x}dx + M\delta_0$ and $d\mu_1(x) = e^{-x}dx$, where $M \geq 0$.

Coherent pairs of measures and Sobolev OP

Jacobi case

1. $d\mu_0(x) = |x-\xi|(1-x)^{\alpha-1}(1+x)^{\beta-1}dx$ and $d\mu_1(x) = (1-x)^{\alpha}(1+x)^{\beta}dx$, where if $|\xi| > 1$ then $\alpha > 0$ and $\beta > 0$, if $\xi = 1$ then $\alpha > -1$ and $\beta > 0$, and if $\xi = -1$ then $\alpha > 0$ and $\beta > -1$.
2. $d\mu_0(x) = (1-x)^{\alpha}(1+x)^{\beta}dx$ and $d\mu_1(x) = \frac{1}{|x-\xi|}(1-x)^{\alpha+1}(1+x)^{\beta+1}dx + M\delta_{\xi}$, where $|\xi| > 1$, $\alpha > -1$ and $\beta > -1$, and $M \geq 0$.
3. $d\mu_0(x) = (1+x)^{\beta-1}dx + M\delta_1$ and $d\mu_1(x) = (1+x)^{\beta}dx$, where $\beta > 0$ and $M \geq 0$.
4. $d\mu_0(x) = (1-x)^{\alpha-1}dx + M\delta_{-1}$ and $d\mu_1(x) = (1-x)^{\alpha}dx$, where $\alpha > 0$ and $M \geq 0$.

Generalized coherent pairs

We can restate (6) as

$$S_n(x;\lambda) + b_{n-1}(\lambda)S_{n-1}(x;\lambda) = P_n(x;d\mu_0) + \widehat{a}_{n-1}P_{n-1}(x;d\mu_0), \quad n \geq 1. \tag{6'}$$

Let $\mathsf{S}_n(x)$ denote the left hand side of (6'). Clearly S_n' can be expanded in terms of $\{P_k(x;d\mu_1)\}$,

$$\mathsf{S}_n'(x) = nP_{n-1}(x;d\mu_1) + \sum_{k=0}^{n-2} d_{k,n}P_k(x;d\mu_1), \qquad d_{k,n} = \frac{\langle \mathsf{S}_n', P_k(x;d\mu_1)\rangle_{d\mu_1}}{\|P_k(x;d\mu_1)\|^2_{d\mu_1}}.$$

We can conclude the following relation between $\{P_n(\cdot;d\mu_0)\}$ and $\{P_n(\cdot;d\mu_1)\}$

$$P_n(x;d\mu_1) + b_{n-1}P_{n-1}(x;d\mu_1) = \frac{P_{n+1}'(x;d\mu_0)}{n+1} + a_n\frac{P_n'(x;d\mu_0)}{n}, \quad n \geq 1. \tag{7}$$

Coherent pairs of measures of the second kind

Let u, v be quasi-definite linear functionals and let $\{P_n^{(0)}\}_{n\geq 0}$ and $\{P_n^{(1)}\}_{n\geq 0}$ be, respectively, the corresponding sequences of monic orthogonal polynomials. We say that the pair $\{u, v\}$ is a *coherent pair of linear functionals of the second kind* (CPLF2K for short) if there exist non-zero constants τ_n such that

$$\frac{1}{n+1} P_{n+1}^{(0)\,\prime}(x) = P_n^{(1)}(x) - \tau_n P_{n-1}^{(1)}(x), \quad n \geq 1. \tag{8}$$

Coherent pairs of measures of the second kind

Theorem

Let u, v be quasi-definite linear functionals. Then, $\{u, v\}$ is a CPLF2K if and only if there exists an admissible pair of polynomials $(A_3, (x)A_2(x))$ with $\deg(A_3) \leq 3$ and $\deg(A_2) = 2$ such that

$$\mathcal{D}v = A_2(x)u \quad \textit{and} \quad v = A_3(x)u. \tag{9}$$

As a consequence, u and v are semiclassical linear functionals of class at most 1, respectively.

Coherent pairs of measures of the second kind

The description of all pairs of coherent pairs of linear functionals of second kind has been recently done by M. Hancco Suni, G. A. Marcato, F. Marcellán and A. Sri Ranga.

As a sake of example, taking into account the Laguerre linear functional satisfies the Pearson equation $D(x\mathcal{L}^{(\alpha)}) = (\alpha+1-x)\mathcal{L}^{(\alpha)}$ you can deduce

1. If $u = \mathcal{L}^{(\alpha)}, \alpha > -1$, is the Laguerre linear functional, then $v = (x-d)\mathcal{L}^{(\alpha+1)} = x(x-d)u, d \leq 0$, i. e. $A_3(x) = x(x-d)$ and $A_2(x) = -x^2 + (\alpha+d+2)x - d(\alpha+1)$. This means that v is semiclassical of class 1.
2. If $v = \mathcal{L}^{(\alpha)}, \alpha > 0$, is the Laguerre linear functional, then $u = (x-c)^{-1}\mathcal{L}^{(\alpha-1)} + M\delta(x-c), c \leq 0$.
3. If u is the truncated Laguerre linear functional, i .e. $\langle u, p\rangle = \int_0^t p(x)x^\alpha e^{-x}dx$, then $\langle v, p\rangle = \int_0^t (t-x)p(x)x^{\alpha+1}e^{-x}dx$, i.e. $A_3(x) = x(t-x), A_2(x) = x^2 - (\alpha+t+2)x + t(\alpha+1)$.

Coherent pairs of measures of the second kind

If $(d\mu_0, d\mu_1)$ is a coherent pair of second kind of positive Borel measures supported on the real line, then we can introduce a Sobolev inner product

$$\langle f, g \rangle_\lambda = \int_a^b f(x)g(x)d\mu_0(x) + \lambda \int_a^b f'(x)g'(x)d\mu_1(x),$$

Theorem

Let $\{Q_n\}_{n\geq 0}$ be the sequence of monic polynomials orthogonal with respect to the above Sobolev inner product. Asssuming $\{d\mu_0, d\mu_1\}$ is a coherent pair of second kind, then there exists a sequence of nonzero real numbers τ_n such that
$P_n(x, d\mu_0) = Q_n(x) - \tau_n Q_{n-1}(x), n \geq 1, \quad (1).$
Conversely, if we have a Sobolev inner product such that (1) holds, then the pair of measures is a coherent pair of the second kind.

OPUC

Given a nontrivial probability measure μ supported on the unit circle T, one defines the OPUC $\{\Phi_n(z;\mu)\}_{n\geq 0}$ as the sequence of monic polynomials orthogonal with respect to the inner product

$$\langle p(z), q(z)\rangle = \int_T p(z)\bar{q}(z^{-1})d\mu(z).$$

They satisfy the forward Szegő recursion:

$$\Phi_{n+1}(z;\mu) = z\Phi_n(z;\mu) - \bar{\alpha}_n\Phi_n^*(z;\mu),$$

where $\alpha_n \in \mathbb{D} := \{z : |z| < 1\}$ and $\Phi_n^*(z) = z^n\overline{\Phi_n}(z^{-1})$.
$\{\alpha_n\}_{n\geq 0}$ is said to be the sequence of Verblunsky (reflection) parameters of the measure μ. Notice that $\bar{\alpha}_n = -\Phi_{n+1}(0;\mu)$. We will denote $A_n^{(\mu)} = ||\Phi_n(z;\mu)||_\mu^2$.

On the other hand, the OPUC satisfy a backward Szegő recursion:

$$z\Phi_n(z;\mu) = \rho_n^{-2}[\Phi_{n+1}(z;\mu) + \bar{\alpha}_n\Phi_{n+1}^*(z;\mu)],$$

with $\rho_n^2 = 1 - |\alpha_n|^2$.

OPUC

To each measure μ supported on the unit circle, we can associate the sequence $\{\alpha_n\}_{n\geq 0}$ of corresponding *Verblunsky coefficients*.

Conversely, given a sequence $\{\alpha_n\}_{n\geq 0}$ of complex numbers with $|\alpha_n| < 1$, there exists a measure μ supported on the unit circle that they constitute the corresponding sequence of Verblunsky parameters. (Favard's theorem).

In the sequel, $\{\varphi_n(z)\}_{n\geq 0}$ will denote the sequence of orthonormal polynomials associated with the measure μ. They can be obtained by using the Gram-Schmidt process for the monomial basis $\{z^n\}_{n\geq 0}$.

The entries of the matrix representation of the multiplication operator in terms of the basis $\{\varphi_n(z)\}_{n\geq 0}$ of the linear space of algebraic polynomials (the GGT matrix) are

$$\mathcal{G}_{i,j} := \left\langle z\varphi_i, \varphi_j \right\rangle.$$

Notice that $\mathcal{G}$ is a upper Hessenberg matrix, whose entries depend on the Verblunsky parameters.

Coherent pairs of measures on the unit circle

Definition

A pair (μ_0, μ_1) of positive measures supported on the unit circle is said to be a coherent pair if the corresponding sequences of monic orthogonal polynomials $\{\Phi_n(\mu_0; z)\}_{n\geq 0}$ and $\{\Phi_n(\mu_1; z)\}_{n\geq 0}$ satisfy the algebraic relation

$$n\Phi_{n-1}(\mu_1; z) = \Phi'_n(\mu_0; z) + \rho_n \Phi'_{n-1}(\mu_0; z), \quad n \geq 2. \tag{10}$$

Here, $\Phi'_n(\mu; z) = d\Phi_n(\mu; z)/dz$.

Coherent pairs of measures on the unit circle

If (μ_0, μ_1) is a coherent pair of positive measures on the unit circle then the following can be stated:

1. If μ_0 is the Lebesgue measure, then the companion measure μ_1 is $d\mu_1(z) = d\mu_0(z)/|z-\alpha|^2$, with $|\alpha| < 1$. That is, μ_1 belongs to the Bernstein-Szegő class.
2. If μ_1 is the Lebesgue measure then the measure μ_0 is such that $d\mu_0(z) = |z-\alpha|^2 d\mu_1(z)$.

The only Bernstein-Szegő measure μ_0 for which (μ_0, μ_1) is a coherent pair is the Lebesgue measure (i.e., μ_0 has to be the Lebesgue measure).

Unfortunately, a full description of all coherent pairs of measures supported on the unit circle is not given and this remains an open problem.

Coherent pairs of measures on the unit circle

An extension of the concept of coherent pair of measures supported on the unit circle has been introduced by Garza-Marcellán-Pinzón and the connection with Sobolev orthogonal polynomials has been discussed. Indeed, they deal with $(1,1)$-coherent pairs of measures such that the corresponding sequences of orthogonal polynomials satisfy

$$n\Phi_{n-1}(\mu_1;z) + \sigma_n\Phi_{n-2}(\mu_1;z) = \Phi_n'(\mu_0;z) + \rho_n\Phi_{n-1}'(\mu_0;z), \quad n \geq 2, \tag{11}$$

with $\rho_n \neq 0$ for $n \geq 2$. The explicit expressions for σ_n and $\rho_n, n \geq 2$, are obtained.

Coherent pairs of measures of the second kind

Definition

A pair of positive measures (μ_0, μ_1) on the unit circle as a coherent pair of measures of the second kind if the following relation holds.

$$\frac{1}{n}\Phi_n'(\mu_0; z) = \Phi_{n-1}(\mu_1; z) - \chi_n \Phi_{n-2}(\mu_1; z), \quad n \geq 2. \tag{12}$$

We will also refer to the constants $\chi_n = \chi_n^{(\mu_0, \mu_1)}$ as the connection coefficients associated with the coherent pair of measures (μ_0, μ_1) of the second kind

Coherent pairs of measures of the second kind

First assumption: The initial assumption is that the measure μ_1 is such that $\mu_1(e^{i\theta}) \in C^1[0,2\pi]$, $\mu_1(e^{i\theta}) \in C^2(0,2\pi)$. Since $\Phi_n'(\mu_0;\zeta) = \frac{d\Phi_n(\mu_0;\zeta)}{d\zeta} = -ie^{-i\theta}\,\frac{d\Phi_n(\mu_0;e^{i\theta})}{d\theta}$ then $\int_0^{2\pi} \frac{d\Phi_n(\mu_0;e^{i\theta})}{d\theta} e^{-ik\theta}\omega_1(\theta)d\theta = 0, \quad 1 \leq k \leq n-2.$

An integration by parts together with $\omega_1(0) = \omega_1(2\pi) < \infty$ yields

$$\int_0^{2\pi} \Phi_n(\mu_0;e^{i\theta})e^{-ik\theta}[-ik\omega_1(\theta) + \omega_1'(\theta)]d\theta = 0, \quad 1 \leq k \leq n-2, \tag{13}$$

and

$$\int_0^{2\pi} \Phi_n(\mu_0;e^{i\theta})e^{-i(n-1)\theta}[-i(n-1)\omega_1(\theta) + \omega_1'(\theta)]d\theta = i\,n\chi_n A_{n-2}^{(\mu_1)}, \quad n \geq 2. \tag{14}$$

Here $\omega_1'(\theta) = d\omega_1(\theta)/d\theta$.

Coherent pairs of measures of the second kind

Second assumption: The orthogonality property suggests that an adequate relation between μ_1 and μ_0 is

$$d\mu_1(e^{i\theta}) = \omega_1(\theta)d\theta = \tau_1|\zeta-\alpha|^2 d\mu_0(e^{i\theta}) = \tau_1\left[1+|\alpha|^2-\alpha e^{-i\theta}-\overline{\alpha}e^{i\theta}\right] d\mu_0(e^{i\theta}), \quad (15)$$

where α is a complex number. The positive number τ_1 can be arbitrary, but throughout in this manuscript we choose its value to be such that μ_0 and μ_1 are non trivial probability measures.

Coherent pairs of measures of the second kind

Since $\omega_1(\theta) \in C^1(0, 2\pi)$ and $\omega_1(0) = \omega_1(2\pi)$,

1. if $|\alpha| \neq 1$ then the function $\mu_0(e^{i\theta})$ needs to be absolutely continuous in $[0, 2\pi]$. Thus, if we set $\int_0^{2\pi} \omega_0(\theta)d\theta = 1$ and $d\mu_0(e^{i\theta}) = \omega_0(\theta)d\theta$ for $[0, 2\pi]$, then $\omega_1(\theta) = \tau_1 \, |e^{i\theta} - \alpha|^2 \omega_0(\theta)$ and $\tau_1^{-1} = \int_0^{2\pi} |e^{i\theta} - \alpha|^2 \omega_0(\theta)d\theta$. Moreover, $\omega_0(0) = \omega_0(2\pi)$;
2. if $\alpha = e^{i\varphi}$ then the measure could have a positive mass of size t $(0 \leq t < 1)$ at $z = \alpha$. However, the function $\mu_0(e^{i\theta})$ needs to be absolutely continuous in $[0, \varphi) \cup (\varphi, 2\pi]$. Thus, if we set $\int_0^{2\pi} \omega_0(\theta)d\theta = 1 - t$ and $\int_0^{2\pi} \ell(e^{i\theta}) \, d\mu_0(e^{i\theta}) = \left[\int_0^{\varphi} \ell(e^{i\theta}) \, \omega_0(\theta)d\theta + \int_{\varphi}^{2\pi} \ell(e^{i\theta}) \, \omega_0(\theta)d\theta \right] + t \, \ell(e^{i\varphi})$, where ℓ is any Laurent polynomial, then $\omega_1(\theta) = \tau_1 \, |e^{i\theta} - \alpha|^2 \omega_0(\theta)$ and $\tau_1^{-1} = \int_0^{2\pi} |e^{i\theta} - \alpha|^2 \omega_0(\theta)d\theta$. Moreover, $\omega_0(0) = \omega_0(2\pi)$ if $\alpha \neq 1$.

Coherent pairs of measures of the second kind

From the orthogonality of $\Phi_n(\mu_0; z)$, for $n \geq 3$ and $1 \leq k \leq n-2$

$$\int_0^{2\pi} \Phi_n(\mu_0; e^{i\theta}) e^{-ik\theta} \omega_1(\theta) d\theta = \tau_1 \int_0^{2\pi} \Phi_n(\mu_0; e^{i\theta}) e^{-ik\theta} |e^{i\theta} - \alpha|^2 d\mu_0(\theta) = 0. \quad (16)$$

Since $\frac{d|e^{i\theta}-\alpha|^2}{d\theta} = i(\alpha e^{-i\theta} - \overline{\alpha} e^{i\theta})$, again from the orthogonality of $\Phi_n(\mu_0; z)$, for $n \geq 3$, and $1 \leq k \leq n-2$,

$$\int_0^{2\pi} \Phi_n(\mu_0; e^{i\theta}) e^{-ik\theta} \frac{d|e^{i\theta} - \alpha|^2}{d\theta} \omega_0(\theta) d\theta = \int_0^{2\pi} \Phi_n(\mu_0; e^{i\theta}) e^{-ik\theta} \frac{d|e^{i\theta} - \alpha|^2}{d\theta} d\mu_0(\theta) = (\quad (17)$$

Coherent pairs of measures of the second kind

For $n \geq 3$,

$$\int_0^{2\pi} \Phi_n(\mu_0; e^{i\theta}) e^{-ik\theta} |e^{i\theta} - \alpha|^2 \omega_0'(\theta) d\theta = 0, \quad 1 \leq k \leq n-2. \tag{18}$$

For the associated connection coefficients we get

$$i\, n\alpha A_n^{(\mu_0)} + \int_0^{2\pi} \Phi_n(\mu_0; e^{i\theta}) e^{-i(n-1)\theta} |e^{i\theta} - \alpha|^2 \omega_0'(\theta) d\theta = i\, \frac{n}{\tau_1} \chi_n A_{n-2}^{(\mu_1)}, \quad n \geq 2. \tag{19}$$

Coherent pairs of measures of the second kind

Third assumption: For the orthogonality property

$$|\zeta-\alpha|^2\omega_0'(\theta) = \left[R(\zeta)+\overline{R(\zeta)}\,\right]\omega_0(\theta), \tag{20}$$

where $R(\zeta) = R(e^{i\theta}) = ue^{i\theta}+v$, with $u, v \in \mathbb{C}$. In the case when $\alpha = e^{i\varphi}$ if the measure μ_0 has a positive mass $t : 0 < t < 1$ at $z = \alpha$ then u and v are also such that $R(\alpha)+\overline{R(\alpha)} = 2\Re(u\alpha)+2\Re(v) = 0$.

When ω_0 satisfies the above relation

$$\chi_n = \tau_1\left[\alpha - i\frac{\overline{u}}{n}\right]\frac{A_n^{(\mu_0)}}{A_{n-2}^{(\mu_1)}}, \quad n \geq 2.$$

Observing that $\omega_0'(\theta) = i\zeta\, d\omega_0(\theta)/d\zeta$, we thus have from (20) the following interesting requirement on the weight function ω_0.

$$i\frac{d\omega_0(\theta)/d\zeta}{\omega_0(\theta)} = \frac{u\zeta^2+(v+\overline{v})\zeta+\overline{u}}{\zeta(\zeta-\alpha)(1-\overline{\alpha}\zeta)}. \tag{21}$$

Coherent pairs of measures of the second kind

Three situations are analyzed separately.

1. If $\alpha = 0$, then
$$i\frac{d\omega_0(\theta)/d\zeta}{\omega_0(\theta)} = u + (v + \overline{v})\frac{1}{\zeta} + \overline{u}\frac{1}{\zeta^2}. \tag{22}$$
2. If $\alpha \neq 0$ but $|\alpha| \neq 1$, then
$$i\frac{d\omega_0(\theta)/d\zeta}{\omega_0(\theta)} = -\frac{1}{\overline{\alpha}}\frac{u\zeta^2 + (v + \overline{v})\zeta + \overline{u}}{\zeta(\zeta - \alpha)(\zeta - 1/\overline{\alpha})}. \tag{23}$$
3. If $\alpha = e^{i\varphi}$, then
$$i\frac{d\omega_0(\theta)/d\zeta}{\omega_0(\theta)} = -\frac{1}{\overline{\alpha}}\frac{u\zeta^2 + (v + \overline{v})\zeta + \overline{u}}{\zeta(\zeta - \alpha)^2}. \tag{24}$$

Coherent pairs of measures of the second kind

Theorem (F. Marcellán, A. Sri Ranga, 2016)

Let $u \in \mathbb{C}$

$$\omega_0(\theta) = \tau_0(u)\, e^{2|u|\,\sin(\theta+\arg u)}, \tag{25}$$

where the positive constant $\tau_0(u)$ *be such that* $\int_0^{2\pi} \omega_0(\theta)d\theta = 1.$
Let the pair of probability measures (μ_0, μ_0) *on the unit circle be given by* $d\mu_0(e^{i\theta}) = \omega_0(\theta)d\theta$*. Then* (μ_0, μ_0) *is a coherent pair of probability measures of the second kind on the unit circle.*
Moreover, for the associated connection coefficients $\chi_n = \chi_n^{(\mu_0,\mu_0)}$ *we have*
$\chi_n = -i\frac{\overline{u}}{n}\frac{A_n^{(\mu_0)}}{A_{n-2}^{(\mu_0)}}, \quad n \geq 1.$

Coherent pairs of measures of the second kind

Theorem (F. Marcellán, A. Sri Ranga, 2016)

Let $u, \alpha \in \mathbb{C}$ and $|\alpha| \neq 1$. Then

$$\omega_0(\theta) = \tau_0(u,\alpha)\, e^{2\Re(u/\overline{\alpha})\arg(1-\alpha e^{-i\theta})}|e^{i\theta}-\alpha|^{-2\Im(u/\overline{\alpha})}, \tag{26}$$

and

$$\omega_1(\theta) = \tau_1(u,\alpha)\, e^{2\Re(u/\overline{\alpha})\arg(1-\alpha e^{-i\theta})}|e^{i\theta}-\alpha|^{2[1-\Im(u/\overline{\alpha})]}, \tag{27}$$

where the positive constants $\tau_0(u,\alpha)$ and $\tau_1(u,\alpha)$ are such that $\int_0^{2\pi}\omega_0(\theta)d\theta = 1$ and $\int_0^{2\pi}\omega_1(\theta)d\theta = 1$, respectively.

Let the pair of probability measures on the unit circle (μ_0, μ_1) be such that $d\mu_0(e^{i\theta}) = \omega_0(\theta)d\theta$ and $d\mu_1(e^{i\theta}) = \omega_1(\theta)d\theta$. Then (μ_0, μ_1) is a coherent pair of probability measures of the second kind on the unit circle. Moreover,

$$\chi_n^{(\mu_0,\mu_1)} = \tau_1\big[\alpha - i\frac{\overline{u}}{n}\big]\frac{A_n^{(\mu_0)}}{A_{n-2}^{(\mu_1)}}, \quad n \geq 1. \tag{28}$$

Coherent pairs of measures of the second kind

Theorem (F. Marcellán, A. Sri Ranga, 2016)

Let $b \in \mathbb{C}$ with $\Re(b) > -1/2$. Then

$$\omega_0(\theta) = \tau_0(b,t)\,(e^{\pi-\theta})^{\Im(b)}\big(\sin^2(\theta/2)\big)^{\Re(b)}, \tag{29}$$

where the positive constant $\tau_0(b,t)$ is such that $\int_0^{2\pi}\omega_0(\theta)d\theta = 1-t$, with $0 \le t < 1$. Let the probability measure μ_0 on the unit circle be given by $d\mu_0(e^{i\theta}) = \omega_0(\theta)d\theta + t\delta(1)$. Moreover, let the probability measure μ_1 be such that $d\mu_1(e^{i\theta}) = \omega_1(\theta)d\theta = \tau_1(b)|e^{i\theta}-1|^2\omega_0(\theta)d\theta$, where $\tau_1(b)$ is such that $\int_0^{2\pi} d\mu_1(e^{i\theta}) = 1$. Then (μ_0,μ_1) is a coherent pair of probability measures of the second kind on the unit circle. Furthermore, the associated connection coefficients satisfy

$$\chi_n^{(\mu_0,\mu_1)} = \tau_1\Big[1+\frac{\overline{b}}{n}\Big] \tag{30}$$

Sobolev OPUC

Given two nontrivial probability measures μ_0 and μ_1 supported on the unit circle and $s > 0$, we denote by $\{\Psi_n(\mu_0, \mu_1, s; z)\}_{n\geq 0}$ the sequence of monic orthogonal polynomials with respect to the Sobolev inner product

$$\langle f, g\rangle_{\mathcal{S}^{(\mu_0,\mu_1,s)}} = \langle f, g\rangle_{\mu_0} + s\,\langle f', g'\rangle_{\mu_1}, \tag{31}$$

where $\langle f, g\rangle_{\mu_0} = \int_{\mathbb{T}} \overline{f(\zeta)} g(\zeta) d\mu_0(\zeta)$ and $\langle f, g\rangle_{\mu_1} = \int_{\mathbb{T}} \overline{f(\zeta)} g(\zeta) d\mu_1(\zeta)$. Clearly, the positive definiteness of the inner product assures the existence of the sequences of orthogonal polynomials $\{\Psi_n(\mu_0, \mu_1, s; z)\}_{n\geq 0}$.

Studies of such Sobolev orthogonal polynomials (or SOPUC) $\{\Psi_n(\mu_0, \mu_1, s; z)\}_{n\geq 0}$ with respect to particular examples of pair of measures (μ_0, μ_1) on the unit circle already appear in the literature.

Sobolev OPUC

Theorem

If (μ_0, μ_1) *is a coherent pair of measures of the second kind on the unit circle, then the sequence of monic orthogonal polynomials,* $\{\Psi_n(z)\}_{n\geq 0}$, *with respect to the Sobolev inner product* $\langle f, g\rangle_{\mathcal{S}^{(\mu_0,\mu_1,s)}}$, *,satisfies*

$\Psi_n(z) - \beta_n \Psi_{n-1}(z) = \Phi_n(\mu_0; z), \quad n \geq 1.$

Morever, $\beta_{n+1} = \frac{q_{n+1}}{p_n - \overline{q}_n\,\beta_n}, \quad n \geq 1, \beta_1 = 0.$

Here

1. $q_n = q_n^{(\mu_0,\mu_1,s)} = s\,n(n-1)A_{n-2}^{(\mu_1)}\,\chi_n, n \geq 1,$
2. $p_n = p_n^{(\mu_0,\mu_1,s)} = A_n^{(\mu_0)} + s\,n^2\left[A_{n-1}^{(\mu_1)} + A_{n-2}^{(\mu_1)}\,|\chi_n|^2\right], n \geq 1.$

   By convention, $A_{-1}^{(\mu_1)} = 0.$

Sobolev OPUC

Let (μ_0, μ_1) be a coherent pair of measures of the second kind on the unit circle. Taking into account $\Psi_{n+1}(z) = \Phi_{n+1}(\mu_0; z) + \sum_{j=1}^{n} a_j^{(n+1)} \Phi_j(\mu_0; z), n \geq 0$, then

Theorem

If $v_n = (a_1^{(n+1)}, a_2^{(n+1)}, \cdots, a_n^{(n+1)})^T$, *then* $T_n v_n = q_{n+1}\, e_n, n \geq 2$, *with*

$$T_n = \begin{pmatrix} p_1 & -q_2 & & & \\ & \ddots & \ddots & \ddots & \\ & & -\bar{q}_{n-1} & p_{n-1} & -q_n \\ & & & -\bar{q}_n & p_n \end{pmatrix}.$$

Sobolev OPUC

Given the coherent pair of measures of the second kind (μ_0, μ_1), let

$$d_n = d_n^{(\mu_0,\mu_1,s)} = \frac{|q_{n+1}|^2}{p_n\, p_{n+1}}, \quad n \geq 1, \tag{32}$$

where $p_n = p_n^{(\mu_0,\mu_1,s)}$ and $q_n = q_n^{(\mu_0,\mu_1,s)}$.

Theorem

The sequence $\{d_n\}_{n\geq 1}$ is a positive chain sequence, i.e., there exists a sequence $\{m_n\}_{n\geq 1}$ such that $0 \leq m_1 < 1, 0 < m_n < 1$ and $(1 - m_n)m_{n+1} = d_n, n \geq 1$.

Theorem (F. Marcellán, A. Sri Ranga, 2016)

Let (μ_0, μ_1) *be the coherent pair of probability measures of the second kind on the unit circle given in Example 3. Let* $\{\Phi_n(\mu_0; z)\}_{n\geq 0}$ *be the sequence of monic orthogonal polynomials with respect to* μ_0 *and let* $\{\Psi_n(\mu_0, \mu_1, s; z)\}_{n\geq 0}$ *be the sequence of monic orthogonal polynomials with respect to the Sobolev inner product* (31). *Then* $\lim_{n\to\infty} \frac{\Psi_n(\mu_0,\mu_1,s;z)}{\Phi_n(\mu_0;z)} = \frac{z}{z-1}$, *in every compact subset of* $\mathbb{C} \setminus \overline{\mathbb{D}}$.

As a direct consequence of the Hurwitz Theorem we deduce that, for n large enough, the zeros of the polynomial $\Psi_n(\mu_0, \mu_1, s; z)$ are located inside the unit circle.

Open problems

- To analyze the behaviour of the connection coefficients β_n in the three cases analyzed above. Notice that in Example 3 they are the ratio between two consecutive continuous dual Hahn polynomials.
- To study the outer ratio asymptotics between the Sobolev orthogonal polynomials and the orthogonal polynomials associated with the measure μ_0 in the Examples 1 and 2.
- The location of zeros of Sobolev orthogonal polynomials in the Example 3 has been recently done by C. F. Bracciali, J. V. da Silva and A. Sri Ranga in JAT 268 (2021) 105604. It is interesting to do the corresponding study in Examples 1 and 2. Their dynamics in terms of the parameter s would have an added value.
- To consider the coherence of second kind when you work with the D_q- operator instead of the derivative.

Some references

1.- M. Hancco Suni, G. A. Marcato, F. Marcellán, A. Sri Ranga, *Coherent sets of measures of the second kind and associated Sobolev orthogonal polynomials. A functional approach* 2022. Submitted.

2.- A. Iserles, P. E. Koch, S. P. Nørsett, J. M. Sanz-Serna, *On polynomials orthogonal with respect to certain Sobolev inner products*, J. Approx. Theory **65** (1991), no. 2, 151-175.

3.- H. G. Meijer, *Determination of all coherent pairs*, J. Approx. Theory **89** (1997), no. 3, 321-343.

4.- A. Branquinho, A. Foulquié, F. Marcellán, M. N. Rebocho, *Coherent pairs of linear functionals on the unit circle,* J. Approx. Theory **153** (2008), 122-137.

5.- M. Hancco Suni, F. Marcellán, A. Sri Ranga, Pastro polynomials and Sobolev type orthogonal polynomials on the unit circle based on a q-differential operator.2022. Submitted.

6.- F. Marcellán, A. Sri Ranga, *Sobolev orthogonal polynomials on the unit circle and coherent pairs of measures of the second kind*, Results Math. **71** (2017) 3-4, 1127 -1149.