Fermi isospectrality for discrete periodic Schrödinger operators

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Operators

- Let Δ be the discrete Laplacian on $\mathbb{Z}^d$:

$$(\Delta u)(n) = \sum_{\|n'-n\|=1} u(n'),$$

where $||n|| = \sum_{i=1}^{d} |n_i|$ for $n = (n_1, n_2, \cdots, n_d) \in \mathbb{Z}^d$.

- Periodic potentials: we say that a function $V : \mathbb{Z}^d \to \mathbb{R}$ is Γ-periodic if for any $\gamma \in \Gamma$, $V(n+\gamma) = V(n)$, where $\Gamma = q_1\mathbb{Z} \oplus q_2\mathbb{Z} \oplus \cdots \oplus q_d\mathbb{Z}$.
- $q_1, q_2, \cdots, q_d$ are distinct prime numbers.
- The discrete periodic Schrödinger operator on $\mathbb{Z}^d$,

$$H_0 = \Delta + V.$$

Eigenvalue problem

- Eigen-equation

$$(\Delta + V)u = \lambda u \tag{1}$$

with Floquet-Bloch boundary condition

$$u(n + q_j \mathbf{e}_j) = e^{2\pi i k_j} u(n), j = 1, 2, \cdots, d. \tag{2}$$

- Fundamental domain W for Γ:

$$W = \{n = (n_1, n_2, \cdots, n_d) \in \mathbb{Z}^d : 0 \leq n_j \leq q_j - 1, j = 1, 2, \cdots, d\}.$$

- Cardinality of W: $Q = q_1 q_2 \cdots q_d$
- By writing out $\Delta + V$ as acting on the Q dimensional space $\{u(n), n \in W\}$, (1) and (2) translate into the eigenvalue problem for a $Q \times Q$ matrix $D_V(k)$, where $k = (k_1, k_2, \cdots, k_d)$.

Floquet Isospectrality

- Two Γ-periodic potentials V and Y are called Floquet isospectral if

$$\sigma(D_V(k)) = \sigma(D_Y(k)), \text{ for any } k \in \mathbb{R}^d.$$

- $(d_1, d_2, \cdots, d_r)$ separable functions: $V = \bigoplus_{j=1}^{r} V_j$.

Questions

Q1. If Y and V are Floquet isospectral, and Y is $(d_1, d_2, \cdots, d_r)$-separable, is V $(d_1, d_2, \cdots, d_r)$-separable?

Q2. Q1 Yes. ($Y = \bigoplus_{j=1}^{r} Y_j$ and $V = \bigoplus_{j=1}^{r} V_j$ are separable) Are the lower dimensional potentials V_j and Y_j Floquet isospectral (up to a constant)?

- Q1 and Q2 have been partially answered by Eskin-Ralston-Trubowitz, Gordon-Kappeler and Kappeler.

Bloch variety

Definition 1

The *Bloch variety* $B(V)$ of $\Delta + V$ consists of all pairs $(k, \lambda) \in \mathbb{C}^{d+1}$ for which there exists a non-zero solution of the equation

$$(\Delta + V)u = \lambda u$$

satisfying the Floquet-Bloch boundary condition

$$u(n + q_j \mathbf{e}_j) = e^{2\pi i k_j} u(n), j = 1, 2, \cdots, d, \text{ and } n \in \mathbb{Z}^d,$$

where $k = (k_1, k_2, \cdots, k_d) \in \mathbb{C}^d$.

Fermi variety

Definition 2

Given $\lambda \in \mathbb{C}$, the Fermi surface (variety) $F_\lambda(V)$ is defined as the level set of the Bloch variety:

$$F_\lambda(V) = \{k : (k, \lambda) \in B(V)\}.$$

$F_\lambda(V)$ is called the Fermi variety of $\Delta + V$ at the energy level λ.

Recall: Main Questions

- V and Y are Floquet isospectral iff $B(Y) = B(V)$ iff $F_\lambda(Y) = F_\lambda(V)$ for every $\lambda \in \mathbb{C}$ (will be discussed later).

Q1. If Y and V are Floquet isospectral (namely $B(Y) = B(V)$ or $F_\lambda(Y) = F_\lambda(V)$ for every λ), and Y is separable, is V separable?

Q2. Furthermore ($Y = \bigoplus_{j=1}^{r} Y_j$ and $V = \bigoplus_{j=1}^{r} V_j$), are the lower dimensional potentials V_j and Y_j Floquet isospectral (namely have the same Bloch varieties)?

Fermi isospectrality vs Floquet isospectrality

- Let V and Y be two Γ-periodic functions. We say V and Y are *Fermi isospectral* if $F_{\lambda_0}(V) = F_{\lambda_0}(Y)$ for some $\lambda_0 \in \mathbb{C}$.
- Floquet isospectrality reformulation: $F_\lambda(V) = F_\lambda(Y)$ for any $\lambda \in \mathbb{C}$ (the Bloch varieties are the same : $B(V) = B(Y)$).

Main results: L-2021 arXiv:2106.03726

$d \geq 3$

1. If Y and V are Fermi isospectral, and Y is $(d_1, d_2, \cdots, d_r)$-separable, then V $(d_1, d_2, \cdots, d_r)$-separable.
2. Futhermore, ($Y = \bigoplus_{j=1}^{r} Y_j$ and $V = \bigoplus_{j=1}^{r} V_j$ are separable) the lower dimensional potentials V_j and Y_j Floquet isospectral (up to a constant).
3. If a potential V and the zero potential are *Fermi isospectral*, then V is zero.

Previous results

(a) For continuous periodic Schrödinger operators, Bättig, Knörrer and Trubowitz proved (2) and (3) when $d = 3$. Their proof is based on the directional compactification, which differs from our approach.

(b) For the discrete case, Kappeler proved that if V and Y are Floquet isospectral, and Y is completely separable, then V is completely separable. Kappeler also answered Q2 affirmatively.

(c) For the continuous case, Eskin, Ralston and Trubowitz proved that if V and Y are Floquet isospectral, and Y is completely separable, then V is completely separable.

(d) For the continuous case, Gordon and Kappeler proved that if V and Y are Floquet isospectral, and Y is separable, then V is separable.

(e) For the continuous case ($d = 2, 3$), Gordon and Kappeler answered Q2 affirmatively.

Bloch and Fermi varieties

- Let $P_V(k,\lambda) = \det(D_V(k) - \lambda I)$ (characteristic function).
- Bloch and Fermi varieties are (principal) analytic sets:

$$B(V) = \{(k,\lambda) \in \mathbb{C}^{d+1} : P_V(k,\lambda) = 0\},$$

and

$$F_\lambda(V) = \{k \in \mathbb{C}^d : P_V(k,\lambda) = 0\}.$$

Based on the irreducibility result (two conjectures [L. GAFA 2022]), we have

- $B(V) = B(Y)$ iff $P_V(k,\lambda) = P_Y(k,\lambda)$ for any k and λ.
- $F_{\lambda_0}(V) = F_{\lambda_0}(Y)$ iff $P_V(k,\lambda_0) = P_Y(k,\lambda_0)$ for any k.

Floquet isospectrality reformulations

- For each $k \in \mathbb{R}^d$, $D_V(k)$ has $Q = q_1 q_2 \cdots q_d$ eigenvalues. Order them in non-decreasing order

$$\lambda_V^1(k) \le \lambda_V^2(k) \le \cdots \le \lambda_V^Q(k).$$

Therefore,

$$P_V(k,\lambda) = \prod_{m=1}^{Q} (\lambda_V^m(k) - \lambda).$$

- Recall (Floquet isospectrality $\sigma(D_V(k)) = \sigma(D_Y(k))$): for any $k \in \mathbb{R}^d$,

$$\lambda_V^m(k) = \lambda_Y^m(k), m = 1, 2, \cdots, Q.$$

- $P_V(k,\lambda) = P_Y(k,\lambda)$
- Floquet isospectrality: $B(V) = B(Y)$

Fermi isospectrality

- For some $\lambda_0 \in \mathbb{C}$, $F_{\lambda_0}(V) = F_{\lambda_0}(Y)$.
- For some λ_0 , $P_V(k, \lambda_0) = P_Y(k, \lambda_0)$ or

$$\prod_{m=1}^{Q} (\lambda_V^m(k) - \lambda_0) = \prod_{m=1}^{Q} (\lambda_Y^m(k) - \lambda_0).$$

Change variables

- Let $z_j = e^{2\pi i k_j}, j = 1, 2, \cdots, d$. $z = (z_1, z_2, \cdots, z_d)$ and $k = (k_1, k_2, \cdots, k_d)$.
- $\mathcal{D}_V(z) = D_V(k)$.
- $\mathcal{P}_V(z, \lambda) = \det(\mathcal{D}_V(z) - \lambda I)$.
- $P_V(k, \lambda) = \mathcal{P}_V(z, \lambda)$.
- $\mathcal{P}_V(z, \lambda)$ is a Laurent polynomial of λ and $z_1, z_2, \cdots, z_d$.

Main ideas of the proof

- Starting point: $\mathcal{P}_V(z,\lambda_0) = \mathcal{P}_Y(z,\lambda_0)$ for some λ_0.
- Study Laurent polynomials $\mathcal{P}_V(z,\lambda_0) = \mathcal{P}_Y(z,\lambda_0)$. Choose proper algebraic curves of $z = (z_1, z_2, \cdots, z_d)$ so that
  - the leading (Laurent) polynomials can be explicitly calculated on those curves
  - the coefficients of the leading (Laurent) polynomials are simple

Choose proper algebraic curves motivated by the Floquet transform

Define

$$\tilde{\mathcal{D}}_V(z) = \mathcal{D}_V(z_1^{q_1}, z_2^{q_2}, \cdots, z_d^{q_d}).$$

Let

$$\rho^j_{n_j} = e^{2\pi i \frac{n_j}{q_j}},$$

where $0 \leq n_j \leq q_j - 1$, $j = 1, 2, \cdots, d$.

Lemma 3

Let $n = (n_1, n_2, \cdots, n_d) \in W$ *and* $n' = (n'_1, n'_2, \cdots, n'_d) \in W$*. Then* $\tilde{\mathcal{D}}_V(z)$ *is unitarily equivalent to* $A + B_V$, *where* A *is a diagonal matrix with entries*

$$A(n; n') = \left(\sum_{j=1}^{d} \left(\rho^j_{n_j} z_j + \frac{1}{\rho^j_{n_j} z_j} \right) \right) \delta_{n,n'}$$

and

$$B_V(n; n') = \hat{V}\left(n_1 - n'_1, n_2 - n'_2, \cdots, n_d - n'_d\right).$$

Examples of algebraic curves

- $$\sum_{j=1}^{d_1} \left(\rho^j_{n_j} z_j + \frac{1}{\rho^j_{n_j} z_j} \right) = \lambda.$$

- $$\sum_{j=1}^{d_1} \rho^j_{n_j} z_j = 0.$$

Lance, Happy Birthday.