A Survey of Left-Definite Operator Theory with Applications to Orthogonal Polynomials

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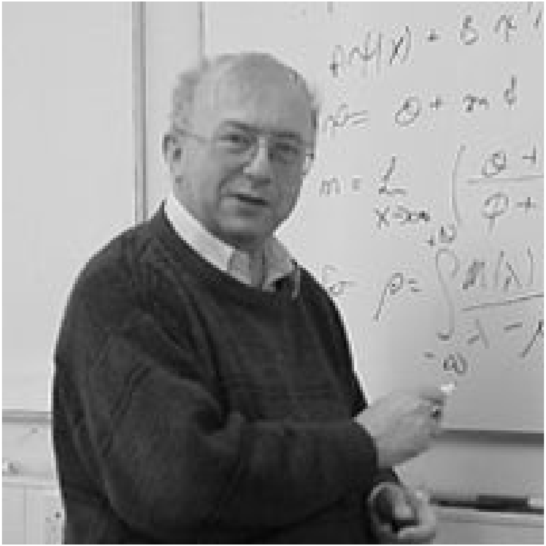
Dedication



Malcolm Brown: 1945-2022

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Original Motivation

Consider for $x \in (a,b)$:

$$\ell[y](x) := -\left(p(x)y'(x)\right)' + q(x)y(x) = \lambda w(x)y(x)$$

where $p, q, w > 0$.

For appropriate functions f, g, we have *Green's formula*:

$$\left(\frac{1}{w}\ell[f], g\right)_{L^2((a,b);w)} = \int_a^b \ell[f]\overline{g}dx = [f,g]\,|_a^b + \left(f, \frac{1}{w}\ell[g]\right)_{L^2((a,b);w)}$$

and *Dirichlet's formula* and *Dirichlet's integral*:

$$\left(\frac{1}{w}\ell[f], g\right)_{L^2((a,b);w)} = \{f,g\}\,|_a^b + \int_a^b \left(pf'\overline{g}' + qf\overline{g}\right)dx.$$

Two "natural" Hilbert spaces to study operators with form $\ell[\cdot]$:

(i) $L^2((a,b);w)$ (the right-definite setting)

(ii) S (the *first* left-definite setting), a certain Sobolev space generated by inner product

$$(f,g)_1 = \int_a^b \left(p(x)f'(x)\overline{g}'(x) + q(x)f(x)\overline{g}(x)\right)dx.$$

Abstract Left-Definite Theory

- Results for this section is found in two Littlejohn-Wellman papers [12] (2002) and [13] (2013).
- **Definition**: Suppose $H = (V, (\cdot, \cdot))$: Hilbert space, $A : \mathcal{D}(A) \subset H \to H$ self-adjoint and bounded below by cI, $c > 0$; that is,

  $$(Ax, x) \geq c(x, x) \quad (x \in \mathcal{D}(A)).$$

  Also suppose V_1 is a linear subspace in V and $(\cdot, \cdot)_1$ is an inner product on $V_1 \times V_1$. Set $H_1 = (V_1, (\cdot, \cdot)_1)$.
- We say that H_1 is a *left-definite space* associated with (H, A) if each of the following five conditions hold:

  (1) H_1 is a Hilbert space;

  (2) $\mathcal{D}(A)$ is a subspace of V_1;

  (3) $\mathcal{D}(A)$ is dense in H_1;

  (4) $(x, x)_1 \geq c(x, x)$ $(x \in V_1)$;

  (5) $(x, y)_1 = (Ax, y)$ $(x \in \mathcal{D}(A), y \in V_1)$.

Observation: If A is self-adjoint and bounded below by cI then for any $r > 0$, A^r is self-adjoint and bounded below by $c^r I$. We can therefore generalize the definition. We note, however, that the literature contained no examples of "higher" left-definite spaces.

Definition: Let $r > 0$. V_r linear subspace in V and $(\cdot,\cdot)_r$ is an inner product on $V_r \times V_r$. Let $H_r = (V_r, (\cdot,\cdot)_r)$. H_r is a r^{th} *left-definite space* associated with (H, A) if:

(1) H_r is a Hilbert space

(2) $\mathcal{D}(A^r)$ is a subspace of V_r

(3) $\mathcal{D}(A^r)$ is dense in H_r

(4) $(x,x)_r \geq c^r(x,x)$ $(x \in V_r)$

(5) $(x,y)_r = (A^r x, y)$ $(x \in \mathcal{D}(A^r),\, y \in V_r)$.

Of course, existence of H_r is certainly in question at this point. In a sense, the most important property is (5).

Theorem Suppose A is a self-adjoint operator in $H = (V, (\cdot, \cdot))$ that is bounded below by cI. Let $r > 0$ and

$$V_r := \mathcal{D}(A^{r/2})$$
$$(x, y)_r := (A^{r/2}x, A^{r/2}y) \quad (x, y \in V_r)$$
$$H_r := (V_r, (\cdot, \cdot)_r).$$

Then H_r is the unique r^{th} left-definite space associated with (H, A). Moreover,

- if A is bounded, then $V = V_r$ and $(\cdot, \cdot)$ and $(\cdot, \cdot)_r$ are equivalent for all $r > 0$.
- if A is unbounded, then V_r is a proper subspace of V and, for $0 < r < s$, V_s is a proper subspace of V_r; moreover, none of the inner products $(\cdot, \cdot)$, $(\cdot, \cdot)_r$, or $(\cdot, \cdot)_s$ are equivalent.
- Moreover, if $\{\phi_n\}$ is a (complete) set of orthogonal eigenfunctions of A in H then they are also a (complete) orthogonal set in each H_r.
- **Heavy** use of the spectral theorem is needed to prove the above theorem.

- These left-definite spaces are *Hilbert scales* (as in the texts of Berezanskiĭ [6] (1968), Albeverio and Kurasov [1] (2000), Maz'ya [15] (2011), and Simon [18] (2005)).
- So let's be clear: **we are not inventing the wheel**!!
- But the applications that we will see are new and they shed further light on the original operator.

Definition: Suppose $H = (V, (\cdot,\cdot))$ is a Hilbert space and A is a self-adjoint operator in H that is bounded below by cI. Let $r > 0$ and $H_r = (V_r, (\cdot,\cdot)_r)$ be the r^{th} left-definite space associated with (H, A). If there exists a self-adjoint operator A_r in H_r that is a restriction of A; i.e.

$$A_r x = Ax$$
$$x \in \mathcal{D}(A_r) \subset \mathcal{D}(A),$$

we call A_r an r^{th} left-definite operator associated with (H, A).

Existence of A_r is also at question at this point.

Theorem Suppose A is a self-adjoint operator in $H = (V, (\cdot,\cdot))$ that is bounded below by cI. Let $r > 0$ and let $H_r = (V_r, (\cdot,\cdot)_r)$ be the r^{th} left-definite space associated with (H, A). Define A_r in H_r by

$$A_r x = Ax \quad (x \in \mathcal{D}(A_r) := V_{r+2}.)$$

Then A_r is the unique left-definite operator associated with (H, A). Moreover, $\sigma(A) = \sigma(A_r)$. In fact, $\sigma_p(A) = \sigma_p(A_r)$ and $\sigma_c(A) = \sigma_c(A_r)$. Furthermore,

- if A is bounded, then $A = A_r$ for all $r > 0$.
- if A is unbounded, then $\mathcal{D}(A_r)$ is a proper subspace of $\mathcal{D}(A)$, and when $0 < r < s$, $\mathcal{D}(A_s)$ is a proper subspace of $\mathcal{D}(A_r)$.
- If $\{\phi_n\}$ is a (complete) set of eigenfunctions of A in H, then they are also a (complete) orthogonal set of eigenfuctions of each A_r.

Example 1: The Laguerre Equation

- For fixed $\alpha > -1$ and $c > 0$, the Laguerre differential expression is defined on $(0, \infty)$ to be:

$$\ell_\alpha[y](x) := -(x^{\alpha+1}e^{-x}y')' + cx^\alpha e^{-x}y = \lambda x^\alpha e^{-x}y$$

- When $\lambda = \lambda_r = r + c$, the r^{th} Laguerre polynomial $y = L_r^\alpha(x)$ is a solution $(r \in \mathbb{N}_0)$.
- $\{L_r^\alpha\}_{r=0}^\infty$ forms a complete orthonormal set in the Hilbert space

$$H := L^2((0, \infty); x^\alpha e^{-x})$$

with inner product

$$(f, g) = \int_0^\infty f(x)\overline{g}(x)x^\alpha e^{-x}dx \quad (f, g \in H).$$

- The right-definite operator $A : \mathcal{D}(A) \subset H \rightarrow H$ defined by

$$Af = \frac{1}{x^{\alpha} e^{-x}} \ell_{\alpha}[f]$$

$$f \in \mathcal{D}(A) = \begin{cases} \Delta & \text{if } \alpha \geq 1 \\ \{f \in \Delta \mid \lim_{x \to 0} x^{\alpha+1} f'(x) = 0\} & \text{if } -1 < \alpha < 1 \end{cases}$$

is self-adjoint in $L^2((0,\infty); x^{\alpha} e^{-x})$ with eigenfunctions $\{L_r^{\alpha}\}_{r=0}^{\infty}$, and discrete (i.e. eigenvalues only) spectrum

$$\sigma(A) = \sigma_p(A) = \{r + c \mid r \in \mathbb{N}_0\};$$

here, Δ is the maximal domain of $\dfrac{1}{x^{\alpha} e^{-x}} \ell_{\alpha}[\cdot]$ in $L^2((0,\infty); x^{\alpha} e^{-x})$ defined by

$$\Delta = \{f \in H | f, f' \in AC_{loc}(0,\infty); \frac{1}{x^{\alpha} e^{-x}} \ell_{\alpha}[f] \in H\},$$

Moreover, for $f \in \mathcal{D}(A)$, we have

$$\begin{aligned}(Af,f) &= \int_0^\infty \left(\mid f'(x) \mid^2 x^{\alpha+1}e^{-x} + c \mid f(x) \mid^2 x^\alpha e^{-x} \right) dx \\ &\geq c \int_0^\infty \mid f(x) \mid^2 x^\alpha e^{-x}\, dx = c(f,f)\end{aligned}$$

so A is bounded below by cI in $L^2((0,\infty);x^\alpha e^{-x})$.

Question: What are the r^{th} left-definite spaces and r^{th} left-definite operators associated with (H,A)?

We can only answer these questions when $r \in \mathbb{N}$.

- The key identity is:

$$\frac{1}{x^{\alpha}e^{-x}}\ell_{\alpha}^{n}[y] = \sum_{j=0}^{n}(-1)^{j}\left(b_{j}(n,c)x^{\alpha+j}e^{-x}y^{(j)}(x)\right)^{(j)}$$

where

$$b_0(n,c) = \begin{cases} 0 & c = 0 \\ c^n & c > 0, \end{cases}$$

$$b_j(n,c) = \begin{cases} S_n^{(j)} & \text{if } c = 0 \\ \sum_{m=0}^{n-1}\binom{n}{m}S_{n-m}^{(j)}c^m & \text{if } c > 0 \end{cases}$$

and $S_n^{(j)}$ is the Stirling number of the second kind.

- Stirling discovered these numbers in 1730 and appear in his calculus book *Methodus Differentialus*. $S_n^{(j)}$ is the number of ways of putting n objects into j non-empty, indistinguishable boxes. (Masanobu Saka, 1782).
- Notice, for $0 \le j \le n$, $b_j(n,c) > 0$ if $c > 0$.

Picture of the cover of Stirling's 1730 book:

Methodus Differentialis:

SIVE

TRACTATUS

DE

SUMMATIONE

ET

INTERPOLATIONE

SERIERUM INFINITARUM.

AUCTORE *JACOBO STIRLING*, R. S. S.

Table of Stirling numbers of the second kind from Stirling's book (e.g. $S_6^{(4)} = 65$)

Tabulam priorem.

1	1	1	1	1	1	1	1	1	&c.
	1	3	7	15	31	63	127	255	&c.
		1	6	25	90	301	966	3025	&c.
			1	10	65	350	1701	7770	&c.
				1	15	140	1050	6951	&c.
					1	21	266	2646	&c.
						1	28	461	&c.
							1	36	&c.
								1	&c.
									&c.

- *Theorem* For $n \in \mathbb{N}$, let $H_n := (V_n, (\cdot,\cdot)_n)$, where

$$\mathcal{D}(A^{n/2}) = V_n = \{f \mid f^{(j)} \in AC_{\text{loc}}(0,\infty) \ (j = 0,1,\ldots,n); \\ f^{(j)} \in L^2((0,\infty); x^{\alpha+j}e^{-x}) \ (j = 0,1,\ldots,n)\}$$

and

$$(f,g)_n := \sum_{j=0}^{n} b_j(n,c) \int_0^\infty f^{(j)}(x)\overline{g}^{(j)}(x)x^{\alpha+j}e^{-x}dx.$$

Then H_n is the n^{th} left-definite space associated with (H, A). and the Laguerre polynomials $\{L_r^\alpha\}_{r=0}^\infty$ form a complete orthogonal set in H_n.

Example 2: The Legendre Equation

- The Legendre polynomials $\{P_r\}_{r=0}^{\infty}$ form a complete orthogonal set of eigenfunctions in $L^2(-1,1)$ of the right-definite self-adjoint operator A defined by

$$A(f(x)) = \ell_{Leg}[f](x) := -(1-x^2)f''(x) + 2xf'(x) + cf(x)$$

for $f \in \mathcal{D}(A)$, defined by

$$\mathcal{D}(A) = \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{loc}(-1,1); \quad f, \ell_{Leg}[f] \in L^2(-1,1); \ \lim_{x\to\pm 1}(1-x^2)f'(x) = 0\};$$

here, c is a fixed positive constant.

- The spectrum of A is discrete and given by

$$\sigma(A) = \sigma_p(A) = \{r(r+1) + c \mid r \in \mathbb{N}_0\}.$$

- Moreover, for $f \in \mathcal{D}(A)$, it is the case that

$$(Af, f) \geq c(f, f).$$

- What are the left-definite spaces $H_n = (V_n, (\cdot, \cdot)_n)$ associated with this Legendre operator?
- To answer, this question, we need first to find the explicit formula for the powers $\ell_{Leg}^n[\cdot]$.
- This turned out to be a cumbersome combinatorial problem...which took me almost three years to solve.

- It turns out that

$$\ell^n_{Leg}[y](x) = \sum_{j=0}^{n} (-1)^j \left(d_j(n,c)(1-x^2)^j y^{(j)}(x) \right)^{(j)},$$

where

$$d_0(n,c) = \begin{cases} 0 & \text{if } c = 0 \\ c^n & \text{if } c > 0, \end{cases}$$

and for $j \in \{1, 2, \ldots, n\}$

$$d_j(n,k) := \begin{cases} PS_n^{(j)} & \text{if } c = 0 \\ \sum_{s=0}^{n-j} \binom{n}{s} PS_{n-s}^{(j)} c^s & \text{if } c > 0, \end{cases}$$

where each $PS_n^{(j)}$ is the *Legendre-Stirling number* defined by

$$PS_n^{(j)} := \sum_{r=1}^{j} (-1)^{r+j} \frac{(2r+1)(r^2+r)^n}{(r+j+1)!(j-r)!} > 0.$$

j/n	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$j=1$	1	1	1	1	1	1	1
$j=2$	–	1	3	7	15	31	63
$j=3$	–	–	1	6	25	90	301
$j=4$	–	–	–	1	10	65	350
$j=5$	–	–	–	–	1	15	140
$j=6$	–	–	–	–	–	1	21
$j=7$	–	–	–	–	–	–	1

Stirling numbers of the second kind (e.g. $S_6^{(4)} = 65$)

j/n	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$j=1$	1	2	4	8	16	32	64
$j=2$	–	1	8	52	320	1936	11648
$j=3$	–	–	1	20	292	3824	47824
$j=4$	–	–	–	1	40	1092	25664
$j=5$	–	–	–	–	1	70	3192
$j=6$	–	–	–	–	–	1	112
$j=7$	–	–	–	–	–	–	1

Legendre-Stirling numbers (e.g. $PS_6^{(4)} = 1092$)

Property	Stirling 2nd Kind	Legendre-Stirling
VRR	$S_n^{(j)} = \sum_{r=j}^{n} S_{r-1}^{(j-1)} j^{n-r}$	$PS_n^{(j)} = \sum_{r=j}^{n} PS_{r-1}^{(j-1)} (j^2+j)^{n-r}$
RGF	$\prod_{r=0}^{j} \frac{1}{1-rx} = \sum_{n=0}^{\infty} S_n^{(j)} x^{n-j}$	$\prod_{r=0}^{j} \frac{1}{1-r(r+1)x} = \sum_{n=0}^{\infty} PS_n^{(j)} x^{n-j}$
TRR	$S_n^{(j)} = S_{n-1}^{(j-1)} + jS_{n-1}^{(j)}$	$PS_n^{(j)} = PS_{n-1}^{(j-1)} + j(j+1)PS_{n-1}^{(j)}$
	$S_n^{(0)} = S_0^{(j)} = 0; S_0^{(0)} = 1$	$PS_n^{(0)} = PS_0^{(j)} = 0; PS_0^{(0)} = 1$
HGF	$x^n = \sum_{j=0}^{n} S_n^{(j)} (x)_j$, where	$x^n = \sum_{j=0}^{n} PS_n^{(j)} \langle x \rangle_j$, where
	$(x)_j = x(x-1)\ldots(x-j+1)$	$\langle x \rangle_j = x(x-2)\ldots(x-(j-1)j))$
1st Kind	$(x)_n = \sum_{j=0}^{n} s_n^{(j)} x^j$	$\langle x \rangle_n = \sum_{j=0}^{n} ps_n^{(j)} x^j$

- Do the Legendre-Stirling numbers $\{PS_n^{(j)}\}$ *count* anything?
- Answer: Yes.
- To see what they count, first consider two copies of each positive integer between 1 and n :

$$1_1, 1_2, 2_1, 2_2, \ldots, n_1, n_2 \quad \text{(two different colors)}.$$

- We now describe two rules on how to fill $j+1$ 'boxes' with the numbers

$$\{1_1, 1_2, 2_1, 2_2, \ldots, n_1, n_2\} :$$

  1. the 'zero box' is the only box that may be empty and it may not contain both copies of any number.
  2. the other j boxes are indistinguishable and each is non-empty; for each such box, the smallest element in that box must contain both copies (or colors) of this smallest number but no other elements can have both copies in that box.
- **Theorem**: *(*Andrews, Littlejohn (2009)) For $n, j \in \mathbb{N}_0$ and $j \leq n$, the Legendre-Stirling number $PS_n^{(j)}$ is the number of different distributions according to the above two rules.

- For each $n \in \mathbb{N}$, the n^{th} left-definite space associated with the pair $(L^2(-1,1), A)$ is given by $H_n = (V_n, (\cdot,\cdot)_n)$, where
-

$$(f,g)_n := \sum_{j=0}^{n} d_j(n,c) \int_{-1}^{1} f^{(j)}(x)\overline{g}^{(j)}(x)(1-x^2)^j dx.$$

and

$$\mathcal{D}(A^{n/2}) = V_n = \{f \mid f \in AC_{loc}^{(n-1)}(-1,1); \\ (1-x^2)^{n/2} f^{(n)} \in L^2(-1,1)\}.$$

- In particular, when $n = 2$, we get a new representation of the domain $\mathcal{D}(A) = V_2$ of the Legendre operator A :

$$\mathcal{D}(A) = \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(-1,1); \\ (1-x^2)^{1/2} f'' \in L^2(-1,1)\}.$$

Example 3: Sobolev-Laguerre Polynomials

- For fixed $k \in \mathbb{N}$ and $c > 0$, the n^{th} degree Laguerre polynomial $y = L_n^{-k}(x)$ (which are *non-classical*) satisfies the second-order Laguerre differential equation

$$\ell_{-k}[y](x) = (n+c)y(x) \quad (n \in \mathbb{N}_0;\, x \in (0,\infty)).$$

- Here

$$\begin{aligned}\ell_{-k}[y](x) &= -xy'' + (-1+k+x)y'(x) + cy(x)\\ &= \frac{1}{x^{-k}e^{-x}}\left(-\left(x^{-k+1}e^{-x}y'(x)\right)' + cx^{-k}e^{-x}y(x)\right)\end{aligned}$$

- The tail-end Laguerre polynomials $\{L_n^{-k}\}_{n=k}^{\infty}$ form a complete orthogonal set in the Hilbert space

$$L^2_{-k}(0,\infty) = L^2((0,\infty); x^{-k}e^{-x})$$

with inner product

$$(f,g) := \int_0^\infty f(x)\overline{g}(x)x^{-k}e^{-x}dx.$$

- This is a consequence of the identity

$$L_n^{-k}(x) = (-1)^k \frac{(n-k)!}{n!} x^k L_{n-k}^k(x) \quad (n \geq k),$$

and the classical orthogonality of $\{L_n^k\}_{n=0}^{\infty}$ with respect to $d\mu = x^k e^{-x} dx$.

- However we note that

$$L_n^{-k} \notin L^2_{-k}(0,\infty) \quad (n = 0, 1, \ldots, k-1).$$

- The right-definite operator

$$A_k : \mathcal{D}(A_k) \subset L^2_{-k}(0,\infty) \to L^2_{-k}(0,\infty)$$

defined by

$$A_k f = \ell_{-k}[f]$$
$$f \in \mathcal{D}(A_k) = \{f : (0,\infty) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(0,\infty); \; f, \ell_{-k}[f] \in L^2_{-k}(0,\infty)\}$$

is self-adjoint with discrete spectrum

$$\sigma(A_k) = \{n + c \mid n \in \mathbb{N}, n \geq k\},$$

and has the tail-end Laguerre polynomials $\{L_n^{-k}\}_{n=k}^{\infty}$ as eigenfunctions.

- Moreover, A_k is bounded below in $L^2_{-k}(0,\infty)$ by cI.
- Consequently, there is a continuum of left-definite spaces $\{H_r = (V_r, (\cdot,\cdot)_r)\}_{r>0}$ and left-definite operators $\{B_r\}_{r>0}$ associated with $(L^2_{-k}(0,\infty), A_k)$.

- Kwon and L. showed that the entire set of Laguerre polynomials $\{L_n^{-k}\}_{n=0}^{\infty}$ are **orthonormal** with respect to the Sobolev inner product

$$\phi_k(f,g) := \sum_{m=0}^{k-1}\sum_{j=0}^{m} B_{m,j}(k)\left[f^{(m)}(0)\overline{g}^{(j)}(0) + f^{(j)}(0)\overline{g}^{(m)}(0)\right] + \int_0^\infty f^{(k)}(x)\overline{g}^{(k)}(x)e^{-x}dx$$

where the numbers $B_{m,j}(k)$ are given by

$$B_{m,j}(k) = \begin{cases} \sum_{p=0}^{j}(-1)^{m+j}\binom{k-1-p}{m-p}\binom{k-1-p}{j-p} & \text{if } 0 \le j < m \le k-1 \\ \frac{1}{2}\sum_{p=0}^{m}\binom{k-1-p}{m-p}^2 & \text{if } 0 \le j = m \le k-1. \end{cases}$$

- In fact, the Laguerre polynomials $\{L_n^{-k}\}_{n=0}^{\infty}$ form a complete orthonormal sequence in the Sobolev space

$$(W_k[0,\infty), \phi_k(\cdot,\cdot)),$$

where

$$W_k[0,\infty) := \{f : [0,\infty) \to \mathbb{C} \mid f, f', \ldots, f^{(k-1)} \in AC_{\text{loc}}[0,\infty); \; f^{(k)} \in L^2((0,\infty); e^{-x})\};$$

- We list a few examples of these (positive definite) inner products:
- $$\phi_1(f,g) = f(0)\overline{g}(0) + \int_0^\infty f'(x)\overline{g}'(x)e^{-x}dx,$$
- $$\begin{aligned}\phi_3(f,g) =& f(0)\overline{g}(0) - 2[f'(0)\overline{g}(0) + f(0)\overline{g}'(0)] \\ & + 5f'(0)\overline{g}'(0) + [f''(0)\overline{g}(0) + f(0)\overline{g}''(0)] \\ & - 3[f''(0)\overline{g}'(0) + f'(0)\overline{g}''(0)] \\ & + 3f''(0)\overline{g}''(0) + \int_0^\infty f'''(x)\overline{g}'''(x)e^{-x}dx.\end{aligned}$$
- In fact, for $k \in \mathbb{N}$ and $f \in W_k[0,\infty)$

$$\phi_k(f,f) = \sum_{r=0}^{k-1}\left|\sum_{j=r}^{k-1}\binom{k-r-1}{j-r}(-1)^{j-r}f^{(j)}(0)\right|^2 + \int_0^\infty \left|f^{(k)}(x)\right|^2 e^{-x}dx \geq 0.$$

- Question: Is there a self-adjoint operator T_k, generated by the Laguerre differential expression $\ell_{-k}[\cdot]$, in $W_k[0,\infty)$ which has the *entire* sequence of Laguerre polynomials $\{L_n^{-k}\}_{n=0}^{\infty}$ as eigenfunctions?. The answer is yes...and it involves left-definite theory.
- Notice that if p is a polynomial of degree $\leq k-1$ and $f \in W_k[0,\infty)$ is such that $f^{(j)}(0) = 0$ for $j = 0, 1, \ldots, k-1$, then

$$\phi_k(f, p) = 0.$$

- This observation leads to the orthogonal decomposition

$$W_k[0,\infty) = W_{k,1}[0,\infty) \oplus W_{k,2}[0,\infty),$$

where $W_{k,1}[0,\infty)$ and $W_{k,2}[0,\infty)$ are closed (so complete) subspaces defined by

$$W_{k,1}[0,\infty) \coloneqq \{f \in W_k[0,\infty) \mid f^{(j)}(0) = 0 \ (j = 0, 1, \ldots, k-1)\},$$

and

$$W_{k,2}[0,\infty) \coloneqq \{f \in W_k[0,\infty) \mid f \text{ is a polynomial of degree } \leq k-1\}.$$

- <u>Plan</u> To construct the self-adjoint operator T_k, generated by $\ell_{-k}[\cdot]$, in $W_k[0,\infty)$ that has the *entire* set of Laguerre polynomials $\{L_n^{-k}\}_{n=0}^{\infty}$ as eigenfunctions, we construct self-adjoint operators $T_{k,1}$ in $W_{k,1}[0,\infty)$ having $\{L_n^{-k}\}_{n=k}^{\infty}$ as eigenfunctions AND $T_{k,2}$ in $W_{k,2}[0,\infty)$ having $\{L_n^{-k}\}_{n=0}^{k-1}$ as eigenfunctions.
- Then we take $T_k = T_{k,1} \oplus T_{k,2}$.
- Constructing $T_{k,2}$ is easy; simply take $T_{k,2}f = \ell_{-k}[f]$ for $f \in \mathcal{D}(T_{k,2}) := W_{k,2}[0,\infty)$. It is easy to show that $T_{k,2}$ is symmetric with respect to $\phi_k(\cdot,\cdot)$ and, since $W_{k,2}[0,\infty)$ is finite-dimensional, $T_{k,2}$ is self-adjoint.
- How do we construct $T_{k,1}$? Remarkably, $T_{k,1}$ is the k^{th} left-definite operator B_k associated with $(L^2_{-k}(0,\infty), A_k)$.
- We now explicitly specify the k^{th} left-definite space $H_k = (V_k, (\cdot,\cdot)_k)$ and the k^{th} left-definite operator B_k associated with $(L_{-k}(0,\infty), A_k)$.

- The k^{th} left-definite Hilbert space

$$H_k = (V_k, (\cdot,\cdot)_k)$$

associated with $(L^2_{-k}(0,\infty), A_k)$ is explicitly given by

$$(f,g)_k = \sum_{j=0}^{k} b_j(k,c) \int_0^\infty f^{(j)}(x)\overline{g}^{(j)}(x)x^{j-k}e^{-x}dx,$$

where, as in Example 1, $b_0(k,c) = c^k$, $b_k(k,c) = 1$, and, for $j = 1,2,\ldots,k-1$,

$$b_j(k,c) = \sum_{m=0}^{k-1} \binom{k}{m} S_{k-m}^{(j)} c^m > 0.$$

- and

$$\begin{aligned}V_k &= \{f:(0,\infty)\to\mathbb{C} \mid f^{(j)} \in AC_{\text{loc}}(0,\infty)\ (j=0,1,\ldots,k-1);\\ &\qquad f^{(j)} \in L^2((0,\infty);x^{j-k}e^{-x})\ (j=0,1,\ldots,k)\}\\ &= \{f:[0,\infty)\to\mathbb{C} \mid f^{(j)} \in AC_{\text{loc}}[0,\infty),\ (j=0,\ldots,k-1);\\ &\qquad f^{(j)}(0)=0\ (j=0,1,\ldots,k-1); f^{(k)} \in L^2((0,\infty);e^{-x})\}\\ &= W_{k,1}[0,\infty).\end{aligned}$$

- Let $f \in H_k = W_{k,1}[0,\infty)$. Then

$$\begin{aligned}\phi_k(f,f) &= \int_0^\infty f^{(k)}(x)\overline{g}^{(k)}(x)e^{-x}dx \\ &\leq \sum_{j=0}^{k} b_j(k,c)\int_0^\infty f^{(j)}(x)\overline{g}^{(j)}(x)x^{j-k}e^{-x}dx \\ &= (f,f)_k.\end{aligned}$$

- So, by the Open Mapping Theorem, the inner products $(\cdot,\cdot)_k$ and $\phi_k(\cdot,\cdot)$ are equivalent on $W_{k,1}[0,\infty) = H_k$.
- We know that the k^{th} left-definite operator $B_k : H_k \to H_k$ for the pair $(L^2_{-k}(0,\infty), A_k)$ defined by

$$\begin{aligned}B_k f &= \ell_{-k}[f] \\ f \in \mathcal{D}(B_k) &= V_{k+2}\end{aligned}$$

is self-adjoint in $H_k = (V_k, (\cdot,\cdot)_k)$. Moreover, the tail-end Laguerre polynomials $\{L_n^{-k}\}_{n=k}^{\infty}$ are a complete set of eigenfunctions in H_k.

- B_k is self-adjoint in $H_k = (V_k, (\cdot,\cdot)_k)$ so B_k is closed in $(W_{k,1}[0,\infty), \phi_k(\cdot,\cdot))$.
- Moreover, it's easy to check that B_k is symmetric in $(W_{k,1}[0,\infty), \phi_k(\cdot,\cdot))$.
- Since a closed symmetric operator which has a complete set of eigenfunctions is necessarily self-adjoint, we see that $T_{k,1} := B_k$ is self-adjoint in $(W_{k,1}[0,\infty), \phi_k(\cdot,\cdot))$.
- To summarize, $T_k = T_{k,1} \oplus T_{k,2}$ is self-adjoint in $(W_k[0,\infty), \phi_k(\cdot,\cdot))$ has the Laguerre polynomials $\{L_n^{-k}\}_{n=0}^{\infty}$ as eigenfunctions, and has spectrum

$$\sigma(T_k) = \sigma_p(T_k) = \{n + c \mid n \in \mathbb{N}_0\}.$$

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