Finite-Rank Perturbations and Applications

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Scalar-valued Aleksandrov–Clark measures

- Consider an analytic self-map b of the unit disc $\mathbb{D}$.
- Corresponding Aleksandrov–Clark family of measures $\{\mu^\alpha\}$ is generated through the Herglotz representation formula

$$\frac{\alpha + b(z)}{\alpha - b(z)} = i \operatorname{Im} \frac{\alpha + b(0)}{\alpha - b(0)} + \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \, d\mu^\alpha(\zeta), \qquad |\alpha| = 1.$$

- Their abs. cont. part is $\frac{d\mu^\alpha}{dm}(\lambda) = \frac{1 - |b(\lambda)|^2}{|\alpha - b(\lambda)|^2}$ for a.e. $\lambda \in \mathbb{T}$.
- Aronsz.–Dono. '57/'65: Let $\alpha, \beta \in \mathbb{T}$, $\alpha \neq \beta$. Then $\mu^\alpha \perp \mu^\beta_{\mathrm{s}}$.
- Nevanlinna '29: Measure μ^α has an atom at $\lambda \in \mathbb{T} \iff$ $b(\lambda) = \alpha$ and b has (bdd) non-tang. derivative at λ (CAD). The point mass is $\mu^\alpha\{\lambda\} = \frac{1}{|b'(\lambda)|}$.

Plan: Generalize to matrix-valued setting, some applications.

Matrix-valued Aleksandrov–Clark measures

- From now on let b denote an $n \times n$ matrix-valued analytic function with $\|b(z)\| < 1$ for all $z \in \mathbb{D}$.
- For any $n \times n$ unitary constant α, the matrix-valued function $H_\alpha(z) := (I + b(z)\alpha^*)(I - b(z)\alpha^*)^{-1}$ is Herglotz.

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- The scalar Herglotz representation formula together with the parallelogram identity yield the matrix-valued analog

$$H_\alpha(z) = i \operatorname{Im} H_\alpha(0) + \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \, d\boldsymbol{\mu}^\alpha(\zeta).$$

- This defines the family of matrix-valued non-negative definite Aleksandrov–Clark measures $\{\boldsymbol{\mu}^\alpha\}$ corresponding to b.

Trace and decomposition of matrix-valued measure $\boldsymbol{\mu}$

- Define the trace by $\mu := \operatorname{tr} \boldsymbol{\mu} = \sum_{k=1}^{n} (\boldsymbol{\mu})_{k,k}$.
- Since $\|A\| \le \operatorname{tr}((A^*A)^{1/2})$, there is a measurable function W from $\mathbb{T}$ to positive definite $n \times n$ matrices with L^∞ entries:

$$d\boldsymbol{\mu}(\lambda) = W(\lambda) d\mu(\lambda).$$

- Weight W is defined wrt μ-a.e., and $\operatorname{tr} W \le 1$ by definition.

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$$d\boldsymbol{\mu} = W w dm + W d\mu_{\mathrm{s}} = d\boldsymbol{\mu}_{\mathrm{ac}} + d\boldsymbol{\mu}_{\mathrm{s}}.$$

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- We also have a weight of the singular part.
- We use the analog notation for the decomposition of $\boldsymbol{\mu}^\alpha$.
- From now on we use $\boldsymbol{\mu} = \boldsymbol{\mu}^I$.

Known for matrix-valued Aleksandrov–Clark measures

Theorem (Mitkovski, IUMJ '11)

The $\operatorname{supp} \mu^\alpha$ *consists of the union of those* $\lambda \in \mathbb{T}$ *at which* $b(z)$ *cannot be analytically continued and those at which* $b(z)$ *is analytically continuable with* $b(\lambda) - \alpha$ *not invertible.*

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Theorem (L.–Martin–Treil, JFA '22)

For unitary α *and a.e.* $\lambda \in \mathbb{T}$ *we have*

$$\frac{d\boldsymbol{\mu}^\alpha}{dm}(\lambda) = \lim (I - \alpha b(z)^*)^{-1}(I - \alpha b(z)^* b(z)\alpha^*)(I - b(z)\alpha^*)^{-1}.$$

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Corollary (Gesztesy–Tsekanovskii, Math. Nachr. '00)

Set $S_\alpha = \left\{\lambda \in \mathbb{T} : \operatorname{tr}\operatorname{Re}(I - b(\lambda)\alpha^*)^{-1} = \infty\right\}$ *forms a carrier of* $\boldsymbol{\mu}_{\mathrm{s}}^\alpha$ *with* $m(S_\alpha) = 0$.
Set $P_\alpha = \left\{\lambda \in \mathbb{T} : \lim (z - \lambda)\operatorname{tr}(I - b(z)\alpha^*)^{-1} \neq 0\right\}$ *is the smallest carrier of* $\boldsymbol{\mu}_{\mathrm{pp}}^\alpha$.

Carrier of Borel measure τ is a Borel set S so that $\tau(\mathbb{R} \setminus S) = 0$.

L.–Treil, APDE '19; L.–Martin–Treil, JFA '21; L.–Treil–Volberg, IMRN '22

- Aronszjan–Donoghue uses 'matrix mutual singularity' and directional support $\mathbf{S}_\alpha(\lambda) := \{\mathbf{e} \in \mathbb{C}^n : \lim b(z)^*\mathbf{e} = \alpha^*\mathbf{e}\}$.
- The set on which 'matrix mutual singularity' does not hold has Hausdorff dimension $\leq n^2 - 1$.

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- Tracial Aronszajn–Donoghue: For singular scalar Borel ν on $\mathbb{T}$, we have $\nu \perp \mu^\alpha$ for Haar a.e. unitary $n \times n$ matrices α.
- Let $\widetilde{\alpha} : \mathbb{R} \to \mathcal{U}_{n\times n}$ be C^1 with sign definite $i\widetilde{\alpha}'(t)\widetilde{\alpha}(t)^{-1}$ for all $t \in \mathbb{R}$. Then for any singular Radon ν on $\mathbb{R}$, we have $\mu^{\widetilde{\alpha}(t)} \perp \nu$ for all $t \in \mathbb{R}$ except possibly countably many.

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- Nevanlinna $\boldsymbol{\mu}^\alpha\{\lambda\} = \lim(1 - z\bar{\lambda})(I - b(z)\alpha^*)^{-1}$
- Connecting point masses with "$b'(z)$" is cumbersome.
- We have Aleksandrov spectral averaging
- We have a good understanding of the adjoint Clark operator (which is the unitary $\Phi^* : L^2(\boldsymbol{\mu}) \to \mathcal{K}_b$ that intertwines the corresponding c.n.u. contraction and the model operator and 'maps' the defect spaces).

Poltoratski's Theorem is open.

Matrix-valued spectral measures of derivative powers

(Bush–L.–Martin, JMAA '22)

Connecting Clark Measures and Defect Spaces

Let T be a differential expression on $L^2(J)$ and $\{\phi_k^{\pm}\}_{k=1}^{n}$ be bases for defect spaces $\mathscr{D}^{\pm}(T)$. Define the analytic $n \times n$ Gram matrix,

$$A(z,w) := \left(\int_J \phi_j^{+}(z,x)\overline{\phi}_k^{\mathsf{sgn}(\mathsf{Im}(w))}(w,x)dx \right)_{j,k=1}^{n},$$

and the Livsic characteristic function,

$$b(z) := \frac{z-i}{z+i}(A(z,i))^{-1}A(z,-i).$$

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Theorem (Aleman–Martin–Ross, JFA '13)

The Aleksandrov–Clark measures of b are the spectral measures of the self-adjoint extensions of T.

For the previous example, we obtained concrete relation between boundary conditions and parameter: $\alpha = \frac{\beta e^{-2a}-1}{\beta - e^{-2a}}$. ...

Defect Index (1,1) AC Example, $-\frac{d^2}{dx^2}$ on $(0,\infty)$

Theorem

Let $|\alpha| = 1$, so $n = 1$. The Radon–Nikodym derivatives of the Clark measures of the self-adjoint extensions of $-\frac{d^2}{dx^2}$ on $(0,\infty)$ are given by

$$\frac{d\mu^{b\alpha^*}}{dm}(s) = \frac{4\mathscr{N}(s)}{\pi\mathscr{D}(s)},$$

where

$$\mathscr{N}(s) := (1 + \mathit{Re}(\alpha))\mathit{Re}(\sqrt{s})(|s| + 1 + 2\mathit{Re}(s^{1/2}e^{-i3\pi/4})),$$

and

$$\mathscr{D}(s) = \left(|s| + 1 + 2\mathit{Re}(\sqrt{s}e^{-i\pi/4})\right)((1-\mathit{Im}(\alpha))(s^3+s)+2s^2-\sqrt{2}\mathit{Re}(\alpha)s^{1/2}+1).$$

Defect Index (2,2) AC Example, $\frac{d^4}{dx^4}$ on $(0,\infty)$

With 2×2 unitary α, and with

$$M := (\gamma^{-1}\alpha^*)^{-1} - b(s), \quad N := I - b^*(s)b(s), \quad \kappa := \pi(1+s^2).$$

The Radon–Nikodym derivatives of the Clark measures of the self-adjoint extensions of $\frac{d^4}{dx^4}$ on $(0,\infty)$ are

$$\left[\frac{d\boldsymbol{\mu}^{b\alpha^*}}{dm}\right]_{1,1} = \frac{|M_{2,2}|^2 N_{1,1} - M_{2,1}\overline{M}_{2,2}N_{1,2} - \overline{M}_{1,2}M_{2,2}N_{2,1} + N_{2,2}|M_{2,1}|^2}{\kappa|\det(M)|^2},$$

$$\left[\frac{d\boldsymbol{\mu}^{b\alpha^*}}{dm}\right]_{1,2} = \frac{\overline{M}_{2,2}(M_{1,1}N_{1,2} - M_{1,2}N_{1,1}) + \overline{M}_{2,1}(M_{1,2}N_{2,1} - M_{1,1}N_{2,2})}{\kappa|\det(M)|^2},$$

$$\left[\frac{d\boldsymbol{\mu}^{b\alpha^*}}{dm}\right]_{2,1} = \frac{\overline{M}_{1,2}(M_{2,1}N_{1,2} - M_{2,2}N_{1,1}) + \overline{M}_{1,1}(M_{2,2}N_{2,1} - M_{2,1}N_{2,2})}{|\det(M)|^2/\kappa},$$

$$\left[\frac{d\boldsymbol{\mu}^{b\alpha^*}}{dm}\right]_{2,2} = \frac{|M_{1,2}|^2 N_{1,1} - M_{1,1}\overline{M}_{1,2}N_{1,2} - \overline{M}_{1,1}M_{1,2}N_{2,1} + N_{2,2}|M_{1,1}|^2}{\kappa|\det(M)|^2}.$$

Singular Boundary Conditions for Sturm–Liouville Operators

(Bush–Frymark–L., arxiv:2011.03388)

Boundary Triples

Again, let T be a closed, symmetric operator that is densely defined on a Hilbert space $\mathscr{H}$ (note that T meets these conditions). $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a **boundary triple** for T^* if $\mathcal{G}$ is a Hilbert space and $\Gamma_0, \Gamma_1 : \mathscr{D}(T^*) \to \mathcal{G}$ are surjective linear maps and the Green's identity

$$\langle T^* f, g\rangle_{\mathcal{H}} - \langle f, T^* g\rangle_{\mathcal{H}} = \langle \Gamma_1 f, \Gamma_0 g\rangle_{\mathcal{G}} - \langle \Gamma_0 f, \Gamma_1 g\rangle_{\mathcal{G}},$$

holds for all $f, g \in \mathscr{D}(T^*)$.

800-page book: Behrndt–Hassi–deSnoo, "Boundary Value Problems, Weyl Functions, and Differential Operators." Monographs in Mathematics 108, Birkhäuser 2020.

Self-adjoint restrictions of $T_{\max}$

Let T_0 denote the operator given by T acting on

$$\mathscr{D}(T_0) = \{f \in \mathscr{D}_{\mathsf{max}}(T) : f \in \ker \Gamma_1\}$$

and T_∞ denote the operator given by T acting on

$$\mathscr{D}(T_\infty) = \{f \in \mathscr{D}_{\mathsf{max}}(T) : f \in \ker \Gamma_0\}.$$

More generally, we have

$$\mathscr{D}(T_\Theta) = \{f \in \mathscr{D}_{\mathsf{max}}(T) : \Theta\Gamma_1 f = \Gamma_0 f\}$$

Example of a Sturm–Liouville operator

For $0 < \alpha, \beta < 1$ consider the Jacobi operator

$$Tf(x) = \ell_{\alpha,\beta}[f](x) = -\frac{1}{(1-x)^{\alpha}(1+x)^{\beta}}[(1-x)^{\alpha+1}(1+x)^{\beta+1}f'(x)]'$$

with

$$\mathscr{D}_{\mathsf{max}}(T) = \{f \in L^2_{\alpha,\beta}(-1,1) | f, f' \in AC_{\mathsf{loc}}(-1,1); Tf \in L^2_{\alpha,\beta}(-1,1)\},$$

where $L^2_{\alpha,\beta}(-1,1) := L^2\left[(-1,1);(1-x)^{\alpha}(1+x)^{\beta}dx\right]$.

- Both endpoints are limit-circle.

It is known (Frymark, JDE 2020) that the mappings

$$\Gamma_0 f := \begin{pmatrix} f^{[0]}(-1) \\ f^{[0]}(1) \end{pmatrix}, \quad \Gamma_1 f := \begin{pmatrix} f^{[1]}(-1) \\ -f^{[1]}(1) \end{pmatrix}, \quad f \in \mathscr{D}_{\mathsf{max}}(T),$$

with

$$\begin{aligned} f^{[0]}(-1) &= \lim_{x \to -1^+} -(1-x)^{\alpha+1}(1+x)^{\beta+1} f'(x), \\ f^{[0]}(1) &= \lim_{x \to 1^-} (1-x)^{\alpha+1}(1+x)^{\beta+1} f'(x), \\ f^{[1]}(-1) &= \lim_{x \to -1^+} -f(x) - \frac{(1+x)f'(x)}{\beta}, \\ f^{[1]}(1) &= \lim_{x \to 1^-} f(x) - \frac{(1-x)f'(x)}{\alpha} \end{aligned}$$

yield a boundary triple. In particular, we have

$$\begin{aligned} \mathscr{D}(T_0) &= \{f \in \mathscr{D}_{\mathsf{max}}(T) \ : \ f^{[1]}(-1) = f^{[1]}(1) = 0\}, \\ \mathscr{D}(T_\infty) &= \{f \in \mathscr{D}_{\mathsf{max}}(T) \ : \ f^{[0]}(-1) = f^{[0]}(1) = 0\}, \\ \mathscr{D}(T_\Theta) &= \{f \in \mathscr{D}_{\mathsf{max}}(T) \ : \ \Theta\Gamma_1 f = \Gamma_0 f\}. \end{aligned}$$

Unperturbed Jacobi Operator

Theorem

The matrix-valued weights of the point masses of T_0 are

$$\boldsymbol{\mu}\{\lambda_n\} = -\frac{\alpha 2^{\alpha+\beta+1}\Gamma(n-\alpha-\beta+1)}{n!\Gamma(1-\alpha)\Gamma(-\beta)} \begin{pmatrix} \prod_{k=0}^{n} \frac{k-\alpha}{k-\beta} & (-1)^n \\ (-1)^n & \prod_{k=0}^{n} \frac{k-\beta}{k-\alpha} \end{pmatrix}.$$

(We verify that the multiplicity of each eigenvalue is one.)

Use Gesztesy–Tsekanovskii, MN '00: $\boldsymbol{\mu}\{\lambda\} = -i \lim \epsilon \mathbf{M}(\lambda + i\epsilon)$.

Theorem

In the spectral representation, the eigenfunction f_n of T_0 corresponding to $\lambda_n = (-n-1)(-n+\alpha+\beta), n \in \mathbb{N}_0$ has the form

$$f_n(x) = \chi_{\{\lambda_n\}}(x) \begin{pmatrix} 1 \\ (-1)^n \prod_{k=0}^{n} \frac{k-\beta}{k-\alpha} \end{pmatrix} \in L^2(\boldsymbol{\mu}).$$

In particular, $L^2(\boldsymbol{\mu}) = \mathit{clos}\,(\mathit{span}\,\{f_n : n \in \mathbb{N}_0\})$.

Let $B^* : \text{Ran}(B) \to \mathbb{C}^2$ be given by

$$B^* f = \begin{pmatrix} f^{[0]}(-1) \\ f^{[0]}(1) \end{pmatrix}.$$

Theorem

Let Θ be a self-adjoint linear relation in $\mathbb{C}^2$ and T_0 be as before, i.e. the self-adjoint Jacobi operator with Neuman boundary conditions. The (form-bounded) singular rank-two perturbation:

$$A_\Theta := T_0 + B\Theta B^*$$

is the same as T_Θ.

Let $\boldsymbol{\mu}^\Theta$ be the spectral measure of T_Θ with respect to B.

Perturbed Jacobi Operator

Eigenvalue $\lambda_n^\Theta \in \rho(T_\infty)$ of T_Θ is **degenerate** if for the $z_n \in \mathbb{R}$ so that $\lambda_n^\Theta = (-z_n - 1)(-z_n + \alpha + \beta)$ we have

$$\Theta_{11} = -\frac{c_4(z_n)}{c_2(z_n)} \quad \text{and} \quad \Theta_{21} = \frac{1}{c_2(z_n)}.$$

(Here $c_2(z) = \frac{\beta 2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(z+1)\Gamma(-z+\alpha+\beta)}$, $c_4(z) = \frac{\beta\Gamma(1-\alpha)\Gamma(\beta)}{\alpha\Gamma(z-\alpha+1)\Gamma(-z+\beta)}$.)

Theorem

Let Θ be 2×2 Hermitian so that all eigenvalues of T_Θ are non-degenerate. In the spectral representation, the eigenvector $f_n^\Theta \in L^2(\boldsymbol{\mu}^\Theta)$ of T_Θ corresponding to λ_n^Θ:

$$f_n^\Theta(x) = \chi_{\{\lambda_n^\Theta\}}(x) \begin{pmatrix} 1 - \Theta_{12} c_2(z_n) \\ \Theta_{11} c_2(z_n) + c_4(z_n) \end{pmatrix}.$$

The space $L^2(\boldsymbol{\mu}^\Theta) = \operatorname{clos\,span}\left\{ f_n^\Theta : n \in \mathbb{N}_0 \right\}$.

Application of L.–Treil, JST '20

Theorem

Let $\Theta_t = \Theta_0 + t\Theta$*, where* Θ*,* Θ_0 *are Hermitian* 2×2 *matrices and* $\Theta > 0$*. For any Hermitian* 2×2 *matrix* $\widetilde{\Theta}$*, the spectra* $\sigma\left(T_{\widetilde{\Theta}}\right) \cap \sigma\left(T_{\Theta_t}\right) = \varnothing$ *for all but possibly countably many* $t \in \mathbb{R}$*.*

Summary

- Properties of matrix-valued Alexandrov–Clark measures
- Applications to derivative powers
- For Jacobi operator, boundary triples and perturbation theory provide complementary information

Thanks!

Generalization of the Aronszjan–Donoghue result

For $d\boldsymbol{\mu} = W d\mu$, meas. matrix-valued fct. Π, and Borel $E \subset \mathbb{T}$

$$\Pi\boldsymbol{\mu}\Pi(E) := \int_E \Pi(z)^*[d\boldsymbol{\mu}(z)]\Pi(z) = \int_E \Pi(z)^* W(z)\Pi(z) d\mu(z).$$

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Definition

Matrix-valued measures $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are said to be *vector mutually singular*, $\boldsymbol{\mu} \perp \boldsymbol{\nu}$, if there exists a measurable function Π with values in the orthogonal projections on $\mathbb{C}^n$ so that

$$\Pi\boldsymbol{\mu}\Pi = \mathbf{O}, \qquad (I - \Pi)\boldsymbol{\nu}(I - \Pi) = \mathbf{O}.$$

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Theorem

We have matrix mutual singularity $\boldsymbol{\mu}^I \perp (I - \alpha^*)\boldsymbol{\mu}_{\mathrm{s}}^\alpha(I - \alpha)$.

The proof is different from the scalar setting, and shorter than that of the analog result for spectral measures. It uses directional support $\mathbf{S}_\alpha(\lambda) := \left\{ \mathbf{e} \in \mathbb{C}^n : \lim_{z \overset{\triangleright}{\to} \lambda \in \mathbb{T}} b(z)^* \mathbf{e} = \alpha^* \mathbf{e} \right\}$ and Poltoratski's Theorem (applied component-wise).

CAD and Carathéodory condition

For $E \leq \mathbb{C}^n$ denote by $P_E : \mathbb{C}^n \to E$ the corresp. orthogonal proj.

Definition

Function b is said to have a CAD at $\lambda \in \mathbb{T}$ on E, if for every $\mathbf{e} \in E$

$$\lim_{z \overset{\triangleright}{\to} \lambda} b(z)\mathbf{e} = \mathbf{e}, \text{ and } \mathrm{CAD}_E b(\lambda) := \lim_{z \overset{\triangleright}{\to} \lambda} P_E b'(z) P_E \text{ exists.}$$

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Function b is said to satisfy the Carathéodory condition in the codirection $\mathbf{x} \in \mathbb{C}^n$ at $\lambda \in \mathbb{T}$ if

$$\liminf_{z \overset{\triangleright}{\to} \lambda} \frac{\|\mathbf{x}\|^2 - \|b(z)^* \mathbf{x}\|^2}{1 - |z|^2} < \infty.$$

CAD and Carathéodory condition

For $E \leq \mathbb{C}^n$ denote by $P_E : \mathbb{C}^n \to E$ the corresp. orthogonal proj.

Definition

Function b is said to have a CAD at $\lambda \in \mathbb{T}$ on E, if for every $\mathbf{e} \in E$

$$\lim_{z \overset{\triangleright}{\to} \lambda} b(z)\mathbf{e} = \mathbf{e}, \text{ and } \mathrm{CAD}_E b(\lambda) := \lim_{z \overset{\triangleright}{\to} \lambda} P_E b'(z) P_E \text{ exists.}$$

Definition

Function b is said to satisfy the Carathéodory condition in the codirection $\mathbf{x} \in \mathbb{C}^n$ at $\lambda \in \mathbb{T}$ if

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Recall $\mathbf{S}_\alpha(\lambda) = \left\{ \mathbf{e} \in \mathbb{C}^n : \lim_{z \overset{\triangleright}{\to} \lambda} b(z)^*\mathbf{e} = \alpha^*\mathbf{e} \right\}$ and consider

$$\mathbf{E}_\alpha(\lambda) := \{\mathbf{x} \in \mathbf{S}_\alpha(\lambda) : b \text{ satisfies Carathéodory cond. at } \lambda \text{ in codir. } \mathbf{x}\}.$$

Characterizing point masses

Theorem

If $b\alpha^*$ has CAD on some subspace E of $\mathbb{C}^n$, then for all $\mathbf{x} \in E$, $P_E b\alpha^*|_E$ satisfies the Carathéodory condition in the codirection $\mathbf{x}$.

Theorem

Function $b\alpha^*$ has CAD at λ on the subspace $\mathbf{E}_\alpha(\lambda)$.

Theorem

We have $\operatorname{Ran} \boldsymbol{\mu}^\alpha(\{\lambda\}) = \mathbf{E}_\alpha(\lambda)$.

Characterizing point masses

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If $b\alpha^*$ *has CAD on some subspace* E *of* $\mathbb{C}^n$*, then for all* $\mathbf{x} \in E$*,* $P_E b\alpha^*|_E$ *satisfies the Carathéodory condition in the codirection* $\mathbf{x}$*.*

The proof uses algebraic manipulations and the triangle inequality.

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We have $\operatorname{Ran} \boldsymbol{\mu}^\alpha(\{\lambda\}) = \mathbf{E}_\alpha(\lambda)$.

For "$\subseteq$", by the first theorem, it suffices to prove that CAD of $b\alpha^*$ at λ exists on the subspace $\operatorname{Ran} \boldsymbol{\mu}^\alpha\{\lambda\}$.