Fixed point theorems and strong asymptotics of multi-level Hermite-Padé polynomials

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Baylor Analysis Fest, Waco: May 23-27, 2022

Outline

- ML HP approximation
- Multiple orthogonality. Statement of the problem
- Analysis of the solution
- References

Nikishin system

Given measures with constant sign $\sigma_\alpha \in \mathcal{M}(\Delta_\alpha), \sigma_\beta \in \mathcal{M}(\Delta_\beta)$ supported on intervals $\Delta_\alpha, \Delta_\beta, \Delta_\alpha \cap \Delta_\beta = \emptyset$, define a third measure by

$$\mathrm{d}\langle \sigma_\alpha, \sigma_\beta \rangle(x) = \hat{\sigma}_\beta(x)\, \mathrm{d}\,\sigma_\alpha(x), \quad \hat{\sigma}_\beta(x) = \int \frac{\mathrm{d}\,\sigma_\beta(t)}{x - t}.$$

Definition

Take a collection $\Delta_j, j = 1, \ldots, m$, of intervals such that $\Delta_j \cap \Delta_{j+1} = \emptyset$, $j = 1, \ldots, m-1$. Let $(\sigma_1, \ldots, \sigma_m)$ be a system of measures with constant sign such that $\mathrm{Co}(\operatorname{supp} \sigma_j) = \Delta_j$, $\sigma_j \in \mathcal{M}(\Delta_j), j = 1, \ldots, m$. We say that $(s_{1,1}, \ldots, s_{1,m}) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$, where

$$s_{1,1} = \sigma_1, \quad s_{1,2} = \langle \sigma_1, \sigma_2 \rangle, \quad \ldots, \quad s_{1,m} = \langle \sigma_1, \langle \sigma_2, \ldots, \sigma_m \rangle \rangle,$$

is the Nikishin system of measures generated by $(\sigma_1, \ldots, \sigma_m)$. The vector $(\widehat{s}_{1,1}, \ldots, \widehat{s}_{1,m})$ is the corresponding Nikishin system of functions.

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ML HP approximation

Definition

Consider $\mathcal{N}(\sigma_1, \ldots, \sigma_m)$. For each $n \in \mathbb{N}$ there exist polynomials $a_{n,0}, a_{n,1}, \ldots, a_{n,m}$, where $\deg a_{n,j} \leq n-1, j = 0, 1, \ldots, m-1$, and $\deg a_{n,m} \leq n$, not all identically equal to zero, called multi-level (ML) Hermite-Padé polynomials of $\mathcal{N}(\sigma_1, \ldots, \sigma_m)$, that verify

$$\mathcal{A}_{n,0}(z) := \left(a_{n,0} + \sum_{k=1}^{m} (-1)^k a_{n,k} \widehat{s}_{1,k}\right)(z) = \mathcal{O}\left(\frac{1}{z^{n+1}}\right), \quad z \to \infty,$$

and for $j = 1, \ldots, m-1$

$$\mathcal{A}_{n,j}(z) := \left((-1)^j a_{n,j} + \sum_{k=j+1}^{m} (-1)^k a_{n,k} \widehat{s}_{j+1,k}\right)(z) = \mathcal{O}\left(\frac{1}{z}\right), \quad z \to \infty,$$

where $s_{j+1,k} = \langle \sigma_{j+1}, \ldots, \sigma_k \rangle$. Set $\mathcal{A}_{n,m} := (-1)^m a_{n,m}$. Take $a_{n,m}$ monic.

Some properties

Lemma

The form $\mathcal{A}_{n,0}$ *has no zero in* $\mathbb{C} \setminus \Delta_1$*. For* $j = 1, \ldots, m$, $\mathcal{A}_{n,j}$ *has exactly* n *zeros in* $\mathbb{C} \setminus \Delta_{j+1}$, $(\Delta_{m+1} = \emptyset)$*, they are all simple and lie in* $\mathring{\Delta}_j$*. If* $Q_{n,j}, j = 1, \ldots, m-1$*, denotes the monic polynomial whose roots are the simple zeros of* $\mathcal{A}_{n,j}$ *in* $\mathring{\Delta}_j$ *then*

$$\frac{\mathcal{A}_{n,j}}{Q_{n,j}} = \mathcal{O}\left(\frac{1}{z^{n+1}}\right) \in \mathcal{H}(\mathbb{C} \setminus \Delta_{j+1}), \qquad z \to \infty.$$

The order of interpolation at infinity is exact.

Lemma

Consider $\mathcal{N}(\sigma_1, \ldots, \sigma_m)$. *For each fixed* $n \in \mathbb{N}$ *and* $j = 0, \ldots, m-1$

$$\int x^\nu Q_{n,j+1}(x) \frac{\mathcal{H}_{n,j+1}(x)\, \mathrm{d}\,\sigma_{j+1}(x)}{Q_{n,j}(x) Q_{n,j+2}(x)} = 0, \qquad \nu = 0, \ldots, n-1 \tag{1}$$

where

$$\mathcal{H}_{n,j}(z) := \frac{Q_{n,j+1}\mathcal{A}_{n,j}}{Q_{n,j}} = \int \frac{Q^2_{n,j+1}(x)}{z-x} \frac{\mathcal{H}_{n,j+1}(x)\, \mathrm{d}\,\sigma_{j+1}(x)}{Q_{n,j}(x) Q_{n,j+2}(x)}. \tag{2}$$

Moreover, the system of monic polynomias $(Q_{n,1}, \ldots, Q_{n,m})$ *of degree* n *which verifies* (1)-(2) *is unique.*

Normalization

Define

$$K_{n,k-1} := \left(\int Q_{n,k}^2(x) \frac{|\mathcal{H}_{n,k}(x)||\,\mathrm{d}\,\sigma_k(x)|}{|Q_{n,k-1}(x) Q_{n,k+1}(x)|} \right)^{-1/2}, \quad k = 1, \ldots, m,$$

$$K_{n,m} := 1, \qquad \kappa_{n,k} := \frac{K_{n,k-1}}{K_{n,k}}, \qquad k = 1, \ldots, m.$$

Set

$$q_{n,k} := \kappa_{n,k} Q_{n,k}, \quad h_{n,k} := K_{n,k}^2 \mathcal{H}_{n,k}, \quad k = 1, \ldots, m, \quad h_{n,0} := K_{n,0}^2 \mathcal{H}_{n,0}.$$

With this notation the expression (1) is equivalent to

$$\int x^\nu Q_{n,k}(x) \frac{|h_{n,k}(x)||\,\mathrm{d}\,\sigma_k(x)|}{|Q_{n,k-1}(x) Q_{n,k+1}(x)|} = 0, \qquad \nu = 0, \ldots, n-1.$$

and $q_{n,k}$ is the n-th orthonormal polynomial.

From (2) it follows that

$$|h_{n,k}(z)| = \left| \int \frac{q_{n,k+1}^2(x)}{z-x} \frac{|h_{n,k+1}(x)||\,\mathrm{d}\,\sigma_{k+1}(x)|}{|Q_{n,k}(x)Q_{n,k+2}(x)|} \right|, \qquad k = 0, \ldots, m-1.$$

Lemma

If $\sigma_k' \neq 0$ a.e. on $\Delta_k = [a_k, b_k], k = 1, \ldots, m$, then

$$\lim_{n\to\infty} |h_{n,k}(z)| = \frac{1}{|\sqrt{(z-b_{k+1})(z-a_{k+1})}|}, \qquad k = 0, \ldots, m-1,$$

uniformly on each compact subset of $\mathbb{C} \setminus \Delta_{k+1}$.

Framework

For each $n \in \mathbb{N}$ and $k = 1, \ldots, m$

$$\int x^{\nu} Q_{n,k}(x) \frac{|h_{n,k}(x)||\,\mathrm{d}\,\sigma_k(x)|}{|Q_{n,k-1}(x) Q_{n,k+1}(x)|} = 0, \qquad \nu = 0, \ldots, n-1.$$

If $\sigma_k' \neq 0$ a.e. on $\Delta_k = [a_k.b_k]$, then

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We want to find the strong asymptotic of the sequences

$$(Q_{n,k})_{n\in\mathbb{N}}, \ (a_{h,k})_{n\in\mathbb{N}}, \ (\mathcal{A}_{n,k})_{n\in\mathbb{N}}, \quad k = 1, \ldots, m.$$

An equilibium problem

Fix $\vec{\Delta} = (\Delta_1, \ldots, \Delta_m)$. Define

$$\mathcal{M}_1(\vec{\Delta}) = \{\vec{\mu} = (\mu_1, \ldots, \mu_m) : \operatorname{supp} \mu_k \subset \Delta_k, |\mu_k| = 1\}.$$

There exist unique $\vec{\lambda} \in \mathcal{M}_1(\vec{\Delta})$ and $\vec{\omega} = (\omega_1, \ldots, \omega_m)$ such that

$$V_{\lambda_k}(x) - \frac{1}{2} V_{\lambda_{k-1}}(x) - \frac{1}{2} V_{\lambda_{k+1}}(x) \begin{cases} \leq \omega_k, & x \in \operatorname{supp}(\lambda_k), \\ \geq \omega_k, & x \in \Delta_k \setminus \operatorname{supp}(\lambda_k), \end{cases}$$

where V_{λ_k} is the logarithmic potential of λ_k $(\lambda_0 \equiv \lambda_{m+1} \equiv 0)$.

(Actually, $\operatorname{supp} \lambda_k = \Delta_k, k = 1, \ldots, m$.)

Weak asymptotic

Theorem

Assume that $\operatorname{supp}\sigma_k = \Delta_k, \sigma_k \in Reg, k = 1, \ldots, m$. *Then*

$$\lim_{n\to\infty} |Q_{n,k}|^{1/n} = \exp(-V_{\lambda_k}), \qquad K \subset \mathbb{C} \setminus \Delta_k,$$

and

$$\lim_{n\to\infty} \kappa_{n,k}^{1/n} = e^{\omega_k}, \qquad k = 1, \ldots, m,$$

where

$$\kappa_{n,k} := \left(\int Q_{n,k}^2(x) \frac{|h_{n,k}(x)||\,d\,\sigma_k(x)|}{|Q_{n,k-1}(x)Q_{n,k+1}(x)|} \right)^{-1/2}, \quad k = 1, \ldots, m.$$

Comparison functions and Szegő functions

Define

$$\Phi_k(z) := e^{-(V_{\lambda_k}+i\widetilde{V}_{\lambda_k})(z)}, \qquad C_k := e^{\omega_k}, \qquad k = 1, 2,$$

where $\widetilde{V}_{\lambda_k}$ is the harmonic conjugate of V_{λ_k} in $\mathbb{C} \setminus \Delta_k$ which equals zero when $z > b_k$.

We write $\mu \in S(\Delta), \Delta = [a, b]$ when $supp\, \mu \subset \Delta$ and

$$\int_\Delta \frac{\ln \mu'(x)\, \mathrm{d}\, x}{\sqrt{(b-x)(x-a)}} > -\infty.$$

Define the Szegő function

$$\mathsf{G}(\mu, z) := \exp\left[\frac{\sqrt{(z-b)(z-a)}}{2\pi}\int_\Delta \frac{\ln(\sqrt{(b-x)(x-a)}\,\mu'(x))}{x-z}\frac{\mathrm{d}\, x}{\sqrt{(b-x)(x-a)}}\right].$$

We have

$$|\mathsf{G}(\mu, x)|^2 = (\sqrt{(b-x)(x-a)}\,\mu'(x))^{-1} \quad \text{a.e. on} \quad \Delta.$$

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The operator T_n

Let

$$\mathcal{Q}_n := \{(Q_1, \ldots, Q_m)) : Q_k \text{ monic pol. of degree } n \text{ and zeros in } \Delta_k\}.$$

Define the operator

$$T_n : \mathcal{Q}_n \longrightarrow \mathcal{Q}_n$$

where $T_n(\tilde{\mathbf{Q}}) = \mathbf{Q}^*, \tilde{\mathbf{Q}} = (\tilde{Q}_1, \ldots, \tilde{Q}_m), \mathbf{Q}^* = (Q_1^*, \ldots, Q_m^*)$ with

$$\int x^\nu Q_k^*(x) \frac{H_k(\tilde{\mathbf{Q}}; x)\, d\,\sigma_k(x)}{\tilde{Q}_{k-1}(x)\tilde{Q}_{k+1}(x)} = 0, \quad \nu = 0, \ldots, n-1, \quad k = 1, \ldots m,$$

$H_m(\tilde{\mathbf{Q}}; x) \equiv 1$, and

$$H_k(\tilde{\mathbf{Q}}; z) = \int \frac{(Q_{k+1}^*(x))^2}{z - x} \frac{H_{k+1}(\tilde{\mathbf{Q}}; x)\, d\,\sigma_{k+1}(x)}{\tilde{Q}_k(x)\tilde{Q}_{k+2}(x)}, \quad k = 1, \ldots, m-1.$$

The components of $\mathbf{Q}^*$ as well as the weights $H_k(\tilde{\mathbf{Q}}; z)$ must be calculated inductively from the value $k = m$ down to $k = 1$.

Notice that $\mathbf{Q}_n := (Q_{n,1}, \ldots, Q_{n,m})$ is a fixed point of the operator T_n and when $\tilde{\mathbf{Q}} = \mathbf{Q}_n$ then

$$H_k(\mathbf{Q}_n; z) = \mathcal{H}_{n,k}(z), \qquad k = 1, \ldots, m.$$

Let $(\tilde{\mathbf{Q}}_n)_{n \geq 0}$ be an arbitrary sequence of vector polynomials (the components of $\tilde{\mathbf{Q}}_n$ are of degree n). Set $T_n(\tilde{\mathbf{Q}}_n) = \mathbf{Q}_n^*$. Define

$$K_{n,k-1}^* := \left(\int (Q_{n,k}^*(x))^2 \frac{|H_k(\tilde{\mathbf{Q}}_k; x)| \, |\mathrm{d}\,\sigma_k(x)|}{|\tilde{Q}_{n,k-1}(x) \tilde{Q}_{n,k+1}(x)|} \right)^{-1/2}, \quad k = 1, \ldots, m$$

$$K_{n,m}^* := 1, \qquad \kappa_{n,k}^* := \frac{K_{n,k-1}^*}{K_{n,k}^*}, \qquad k = 1, \ldots, m.$$

Set

$$q_{n,k}^* := \kappa_{n,k}^* Q_{n,k}^*, \quad h_k(\tilde{\mathbf{Q}}_n; x) := (K_{n,k}^*)^2 H_k(\tilde{\mathbf{Q}}_n; x), \quad k = 1, \ldots, m.$$

If $\sigma_k' > 0$ a.e. on $\Delta_k, k = 1, \ldots, m$, then

$$\lim_{n \to \infty} |h_k(\tilde{\mathbf{Q}}_n; z)| = \frac{1}{|\sqrt{(z - b_{k+1})(z - a_{k+1})}|}, \quad k = 0, \ldots, m-1,$$

uniformly on each compact subset of $\mathbb{C} \setminus \Delta_{k+1}$.

Theorem

Assume that $|\sigma_k| \in S(\Delta_k), k = 1, \ldots, m$. *Let* $(\tilde{\mathbf{Q}}_n)_{n\in\mathbb{N}}$ *be such that*

$$\lim_{n\to\infty} \frac{\tilde{Q}_{n,k}}{\Phi_k^n} = f_k > 0, \qquad \textit{unif. on } \Delta_{k-1} \cup \Delta_{k+1} \qquad k = 1, \ldots, m.$$

Then

$$\lim_{n\to\infty} \frac{Q^*_{n,k}(z)}{\Phi_k^n(z)} = \frac{G_k(z)}{G_k(\infty)}, \qquad K \subset \overline{\mathbb{C}} \setminus \Delta_k, \qquad k = 1, \ldots, m,$$

where G_k *is the Szegő function on* $\overline{\mathbb{C}} \setminus \Delta_k$ *with boundary conditions*

$$|G_k(x)|^2 = \frac{\sqrt{|x-b_{k+1}||x-a_{k+1}|}\,f_{k-1}(x)f_{k+1}(x)}{\sqrt{(b_k-x)(x-a_k)}\,|\sigma_k'(x)|}, \quad \textit{a.e. on } \Delta_k,$$

and

$$f_0(x) \equiv f_{m+1}(x) \equiv \sqrt{|x-b_{m+1}||x-a_{m+1}|} \equiv 1$$

A boundary value problem

Set $\mathbf{w} = (w_1, \ldots, w_m)$, where

$$w_k(x) = \frac{\sqrt{(b_k - x)(x - a_k)}|\sigma_k'(x)|}{\sqrt{|x - b_{k+1}||x - a_{k+1}|}}, \qquad x \in \Delta_k.$$

Let $C^+_{\vec{\Delta}}$ be the metric space of all $\mathbf{g} = (g_1, \ldots, g_m)$, where g_k is continuous and positive on $\Delta_{k-1} \cup \Delta_{k+1}$ endowed with the distance

$$d(\mathbf{f}, \mathbf{g}) = \max\{\| \ln(g_k/f_k)\|_{\Delta_{k-1}\cup\Delta_{k+1}} : k = 1, \ldots, m\}.$$

This metric space is complete.
Let

$$\mathcal{G} := \{\mathbf{G} = (G_1, \ldots, G_m) : G_k \text{ Szegő function in } \overline{\mathbb{C}} \setminus \Delta_k\}$$

Define $T : C^+_{\vec{\Delta}} \longrightarrow \mathbb{C}^+_{\vec{\Delta}} \subset \mathcal{G}$ where $T(\tilde{g}_1, \ldots, \tilde{g}_m) = (g_1^*, \ldots, g_m^*)$ with

$$|g_k^*(x)|^2 = \frac{\tilde{g}_{k-1}(x)\tilde{g}_{k+1}(x)}{w_k(x)}, \quad \text{a.e. on } \Delta_k,$$

and $\tilde{g}_0(x) \equiv \tilde{g}_{m+1}(x) \equiv 1$.

Existence of unique fixed point

Theorem

Assume that $|\sigma_k| \in S(\Delta_k), k = 1, \ldots, m$ *and let* $N = \lceil m/2 \rceil$. *Then*

$$d(T^N(\mathbf{f}), T^N(\mathbf{g})) \leq \gamma_m d(\mathbf{f}, \mathbf{g}), \quad \gamma_m = \begin{cases} (2^N - 2)/2^N, & m \ odd, \\ (2^N - 1)/2^N, & m \ even. \end{cases}$$

Consequently, T^N *has a unique fixed point and so does* T.

Main theorem

Theorem

Let $|\sigma_k| \in S(\Delta_k), k = 1, \ldots, m$ *and* $\mathbf{G} = (G_1, \ldots, G_m)$ *be the fixed point of* T. *Then, for* $k = 1, \ldots, m$,

$$\lim_{n\to\infty} \frac{Q_{n,k}(z)}{\Phi_k^n(z)} = \frac{G_k(z)}{G_k(\infty)}, \qquad K \subset \overline{\mathbb{C}} \setminus \Delta_k$$

and

$$\lim_{n\to\infty} \frac{\kappa_{n,k}}{C_k^n} = \frac{1}{\sqrt{2\pi}} G_k(\infty),$$

where

$$\kappa_{n,k} = \left(\int \frac{|Q_{n,k}(x)|^2 |h_{n,k}(x)| |\,\mathrm{d}\,\sigma_k(x)|}{|Q_{n,k-1}(x) Q_{n,k+1}(x)|} \right)^{-1/2}.$$

Since $a_{n,m} = Q_{n,m}$, we have

$$\lim_{n\to\infty} \frac{a_{n,m}(z)}{\Phi_m^n(z)} = \frac{G_m(z)}{G_m(\infty)}, \qquad K \subset \overline{\mathbb{C}} \setminus \Delta_m.$$

On the other hand, for $k = 0, \ldots, m-1$,

$$\lim_{n\to\infty} \frac{a_{n,k}(z)}{\Phi_m^{n-1}(z)} = \Phi_m(z) \lim_{n\to\infty} \frac{a_{n,k}(z)}{a_{n,m}(z)} \frac{a_{n,m}(z)}{\Phi_m^n(z)} =$$

$$\Phi_m(z)\widehat{s}_{m,k+1}(z)\frac{G_m(z)}{G_m(\infty)}, \qquad K \subset \overline{\mathbb{C}} \setminus \Delta_m.$$

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Thank you