Spectral Exploration of Exceptional Laguerre Operators

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History

- 2009: D. Gómez-Ullate, N. Kamran, R. Milson in "An extended class of orthogonal polynomials defined by a Sturm-Liouville problem" and "An extension of Bochner's problem: exceptional invariant subspaces"
- 2013-2016:
  - M. J. Atia,, L. L. Littlejohn, J. Stewart Kelly in "Spectral theory of X_1-Laguerre polynomials"
  - C. Liaw, L. Littlejohn, J. Stewart Kelly in "Spectral analysis for the exceptional X_m-Jacobi Equation"
  - C. Liaw, L. Littlejohn, J. Stewart Kelly, Q. Wicks in "A spectral study of the second-order exceptional X_1-Jacobi differential expression and a related non-classical Jacobi differential expression"
  - C. Liaw, L. Littlejohn, R. Milson, J. Stewart Kelly in "The spectral analysis of three families of exceptional Laguerre polynomials"
- 2021: D. Gómez-Ullate, Y. Grandati, R. Milson in "Spectral Theory of Exceptional Hermite Polynomials"

Introduction

Question

For general Laguerre exceptional orthogonal polynomial (XOP) families, can we exploit properties of the Darboux transform to determine self-adjoint expressions and the spectrum?

1. Begin with two linearly independent solutions associated with the classical orthogonal polynomial (COP) operator
2. Apply the Darboux transform to these solutions
3. Build a boundary triple for a general XOP operator
4. Calculate the Weyl m-function for self-adjoint extensions associated with the XOP operator
5. Extract information concerning self-adjoint expressions and spectrum

Type I Laguerre XOP

- Recall Laguerre COP operator ℓ^α has rational factorization

$$-\ell^\alpha = B^{I,\alpha}_m \circ A^{I,\alpha}_m + \alpha + m + 1,$$

where

$$A^{I,\alpha}_m[f] := L^\alpha_m(-x) f'(x) - L^{\alpha+1}_m(-x) f(x), \text{ and}$$
$$B^{I,\alpha}_m[f] := \frac{x f'(x) + (1+\alpha) f(x)}{L^\alpha_m(-x)}.$$

- This Type I Laguerre XOP operator may be defined by

$$\ell^{I,\alpha}_m = -\left(A^{I,\alpha-1}_m \circ B^{I,\alpha-1}_m + \alpha + m\right).$$

- The Type I Laguerre polynomials $\{L^{I,\alpha}_{m,n}(x)\}_{n=m}^\infty$ are orthogonal on $(0,\infty)$ with respect to

$$W^{I,\alpha}_m(x) := \frac{x^\alpha e^{-x}}{\left(L^{\alpha-1}_m(-x)\right)^2}.$$

Type I Laguerre XOP

Theorem 3.5 of Liaw, Littlejohn, S, Milson

Suppose $0 < \alpha < 1$. The operator

$$T_m^{I,\alpha} : \mathcal{D}(T_m^{I,\alpha}) \subset L^2\left((0,\infty); W_m^{I,\alpha}\right) \to L^2\left((0,\infty); W_m^{I,\alpha}\right) \,,$$

defined by

$$T_m^{I,\alpha}\; f = \ell_m^{I,\alpha}[f]$$

$$f \in \mathcal{D}(T_m^{I\alpha}) := \left\{ f \in \Delta_m^{I,\alpha} \mid \lim_{x \to 0^+} x^{\alpha+1} f'(x) = 0 \right\},$$

is self-adjoint in $L^2\left((0,\infty); W_m^{I,\alpha}\right)$ and has the Type I exceptional Laguerre polynomials $\left\{L_{m,n}^{I,\alpha}\right\}_{n=m}^{\infty}$ as eigenfunctions. Moreover, the spectrum of $T_m^{I,\alpha}$ consists only of eigenvalues and is given by

$$\sigma(T_m^{I,\alpha}\;) = \mathbb{N}_0 \,.$$

Type I Laguerre XOP

- The maximal domain associated with $\ell_m^{I,\alpha}$ in the Hilbert space $L^2\left((0,\infty), W_m^{I,\alpha}\right)$ is defined as

$$\Delta_m^{I,\alpha} := \left\{ f : (0,\infty) \to \mathbb{C} \ : \ f, f' \in AC_{\text{loc}}(0,\infty); f, \ell_m^{I,\alpha}[f] \in L^2\left((0,\infty), W_m^{I,\alpha}\right) \right\}$$

- The sesquilinear form, for $f, g \in \Delta_m^{I,\alpha}$ and $x \in (0,\infty)$, is given by

$$[f,g]_m^{I,\alpha}(x) := \frac{x^{\alpha+1}e^{-x}}{[L_m^{\alpha-1}(-x)]^2}\left[f(x)g'(x) - f'(x)g(x)\right].$$

- Define quasi-derivatives

$$\Gamma_0(f(0)) := [f, x^{-\alpha}]_L(0) = \lim_{x\to 0^+} -\frac{\alpha f(x) + x f'(x)}{[L_m^{\alpha-1}(0)]^2},$$

$$\Gamma_1(f(0)) := \left[f, \frac{[L_m^{\alpha-1}(0)]^2}{\alpha}\right]_L(0) = \lim_{x\to 0^+} \frac{x^{\alpha+1}}{\alpha} f'(x).$$

Type I Laguerre XOP

- Intertwining property: $A_m^{I,\alpha-1}[-\ell^{\alpha-1}] = \ell_m^{I,\alpha}[A_m^{I,\alpha-1}]$
- Two linearly independent solutions of the classical eigenvalue problem $(\ell^{\alpha-1} - \lambda)h = 0$

$$h^{\alpha-1}(x,\lambda) = M(-\lambda,\alpha,x) = {}_1F_1(-\lambda,\alpha,x) \qquad \text{(for } \alpha+1 \notin -\mathbb{N}_0\text{)},$$
$$\widetilde{h}^{\alpha-1}(x,\lambda) = x^{-\alpha+1}M(1-\lambda-\alpha, 2-\alpha, x) \qquad \text{(for } \alpha+1 \notin \mathbb{N} > 1\text{)}.$$

- Define

$$\Phi_m^{\alpha}(x,\lambda) := C_m^{\alpha}(\lambda)A^{I,\alpha-1}[h^{\alpha-1}(x,\lambda)]$$
$$\Psi_m^{\alpha}(x,\lambda) := D_m^{\alpha}(\lambda)A^{I,\alpha-1}[\widetilde{h}^{\alpha-1}(x,\lambda)]$$

Type I Laguerre XOP

where

$$C_m^{\alpha}(\lambda) := -\frac{\Gamma(m+\alpha)}{(\lambda+m+\alpha)\Gamma(\alpha)\Gamma(m+1)} \text{ and } D_m^{\alpha(\lambda)} := -\frac{\Gamma(m+1)\Gamma(\alpha)}{(1-\alpha)\Gamma(m+\alpha)}$$

ensure the initial conditions

$$\begin{pmatrix} \Gamma_0(\Phi_m^{\alpha}(0,\lambda)) & \Gamma_0(\Psi_m^{\alpha}(0,\lambda)) \\ \Gamma_1(\Phi_m^{\alpha}(0,\lambda)) & \Gamma_1(\Psi_m^{\alpha}(0,\lambda)) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

are satisfied.

By Proposition 6.4.9 of Behrndt, Hassi, and de Snoo, $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is a boundary triple for $\Delta_m^{I,\alpha}$.

Type I Laguerre XOP

Constructing the Weyl m-function

- $$\operatorname{dom}(L_0) = \left\{ f \in \Delta_m^{I,\alpha} \;:\; \Gamma_1(f) = 0 \right\}, \text{ and}$$
  $$\operatorname{dom}(L_\infty) = \left\{ f \in \Delta_m^{I,\alpha} \;:\; \Gamma_0(f) = 0 \right\}$$

- In the case of the Laguerre COP operator, the general deficiency element X is the Tricomi confluent hypergeometric function

$$X(-\lambda,\alpha,z) = \frac{\Gamma(1-\alpha)}{\Gamma(1-\alpha-\lambda)} M(-\lambda,\alpha,z) + \frac{\Gamma(\alpha-1)}{\Gamma(-\lambda)} z^{-\alpha+1} M(1-\lambda-\alpha, 2-\alpha, z).$$

- For the Laguerre XOP operator,

$$\begin{aligned}\chi(\lambda,\alpha,z) &= -A^{I,\alpha-1}[X(-\lambda,\alpha,z)] \\ &= \frac{\Gamma(1-\alpha)}{C_m^\alpha(\lambda)\Gamma(1-\alpha-\lambda)} \Phi_m^\alpha(z) + \frac{\Gamma(\alpha-1)}{D_m^\alpha \Gamma(-\lambda)} \Psi_m^\alpha(z)\end{aligned}$$

Type I Laguerre XOP

Constructing the L_∞ Weyl m-function:

- Weyl m-function for the extension L_∞ is given by

$$M_\infty(\lambda) = \frac{\Gamma_1(\chi(\lambda,\alpha,0))}{\Gamma_0(\chi(\lambda,\alpha,0))} = -\left[L_m^{\alpha-1}(0)\right]^2 \frac{\Gamma(\alpha)\Gamma(1-\alpha-\lambda)}{(\lambda+m+\alpha)\Gamma(-\lambda)\Gamma(1-\alpha)}.$$

- The spectrum of the self-adjoint operator L_∞ are those points which are poles of $M_\infty(\lambda)$. The Gamma function has simple poles at zero and the negative integers. Therefore, $\sigma(L_\infty) = (-m-\alpha) \cup \{n+1-\alpha\}_{n\in\mathbb{N}_0}$.

Constructing the L_0 Weyl m-function:

- Weyl m-function for the extension L_0 is given by

$$M_0(\lambda) = \frac{-1}{M_\infty(\lambda)} = \frac{-(\lambda+m+\alpha)\Gamma(-\lambda)\Gamma(1-\alpha)}{\left[L_m^{\alpha-1}(0)\right]^2 \Gamma(\alpha)\Gamma(1-\alpha-\lambda)}.$$

- Therefore, $\sigma(L_0) = \{n\}_{n\in\mathbb{N}_0}$.

Type I Laguerre XOP

In fact, the m-function for all self-adjoint extensions can now be written down as well. Let $\tau \in \mathbb{R} \cup \{\infty\}$ and L_τ refer to the self-adjoint operator acting via $\ell_m^{I,\alpha}$ on

$$\operatorname{dom}(L_\tau) := \left\{ f \in \Delta_m^{I,\alpha} \middle| f \in \ker\left(\Gamma_1 - \tau\Gamma_0\right) \right\} = \left\{ f \in \Delta_m^{I,\alpha} \middle| \Gamma_1(f(0)) = \tau\Gamma_0(f(0)) \right\}.$$

The corresponding Weyl m-function for $\lambda \in \rho(L_\infty) \cup \rho(L_\tau)$ will be

$$\begin{aligned} M_\tau(\lambda) &= \frac{1 + \tau M_\infty(\lambda)}{\tau - M_\infty(\lambda)} \\ &= \frac{(\lambda + m + \alpha)\Gamma(-\lambda)\Gamma(1-\alpha) - \tau\left[L_m^{\alpha-1}(0)\right]^2 \Gamma(\alpha)\Gamma(1-\alpha-\lambda)}{\tau(\lambda + m + \alpha)\Gamma(-\lambda)\Gamma(1-\alpha) + \left[L_m^{\alpha-1}(0)\right]^2 \Gamma(\alpha)\Gamma(1-\alpha-\lambda)}. \end{aligned}$$

Hence, if $\tau \neq \{0, \infty\}$, L_τ will have eigenvalues precisely when

$$\tau(\lambda + m + \alpha)\Gamma(-\lambda)\Gamma(1-\alpha) = -\left[L_m^{\alpha-1}(0)\right]^2 \Gamma(\alpha)\Gamma(1-\alpha-\lambda).$$

General XOP Framework

To produce a Laguerre XOP family, a general (but finite) sequence of Darboux transformations applied to a function $f(x)$ may be written as taking the Wronskian of $h(x)$ with a set of "seed" functions:

$$
\begin{aligned}
f_j(x) &= L^{(\alpha)}_{n_j}(x), & j &= 1, \dots, r_1, \\
f_{r_1+j}(x) &= e^x L^{(\alpha)}_{m_j}(-x), & j &= 1, \dots, r_2, \\
f_{r_1+r_2+j}(x) &= x^{-\alpha} L^{(-\alpha)}_{m'_j}(x), & j &= 1, \dots, r_3, \\
f_{r_1+r_2+r_3+j}(x) &= e^x x^{-\alpha} L^{(-\alpha)}_{n'_j}(-x), & j &= 1, \dots, r_4,
\end{aligned}
$$

where $r_1 + r_2 + r_3 + r_4 = r$, $n_1 > \cdots > n_{r_1} \geq 0$, $m_1 > \cdots > m_{r_2} \geq 0$, $n'_1 > \cdots > n'_{r_1} \geq 0$ and $m'_1 > \cdots > m'_{r_4} \geq 0$.

The degrees of these "seed" functions may be described via Maya diagrams:

$$
\begin{aligned}
M_1 &= \left(n'_1, \dots, n'_{r_4} | n_1, \dots, n_{r_1}\right), \\
M_2 &= \left(m'_1, \dots, m'_{r_3} | m_1, \dots, m_{r_2}\right).
\end{aligned}
$$

Example

Consider Maya Diagrams $M_1 = (\emptyset | 3, 2)$ and $M_2 = (1, 0 | \emptyset)$.

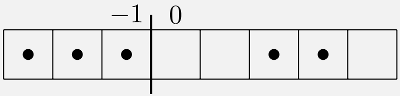

Figure: Diagram for M_1

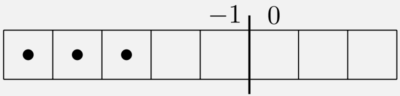

Figure: Diagram for M_2

It follows that:

- $n = (3, 2)$, $m = \emptyset$,
  $n' = \emptyset$, $m' = (1, 0)$;
- $t_1 = 0$, $t_2 = 2$;
- $M_1 + t_1 = (\emptyset | 3, 2)$,
  $M_2 + t_2 = (\emptyset | \emptyset)$; and
- $r_1 = r_3 = 2$,
  $r_2 = r_4 = 0$;

General XOP Framework

Two linearly independent (at $x = 0$) solutions to the Laguerre COP equation:

$$h^{\alpha}(x,\lambda) := M(-\lambda, \alpha+1, x) = {}_1F_1(-\lambda, \alpha+1, x)$$

for $\alpha + 1 \notin -\mathbb{N}_0$,

and

$$\widetilde{h}^{\alpha}(x,\lambda) := x^{-\alpha} M(-\lambda-\alpha, 1-\alpha, x)$$

for $\alpha + 1 \notin \mathbb{N} > 1$.

Using $h^{\alpha}(x,\lambda)$ and $\widetilde{h}^{\alpha}(x,\lambda)$ and multiplying by the appropriate prefactor yields

$$\phi^{\alpha}_{M_1,M_2}(x,\lambda) := e^{-(r_2+r_4)x} x^{(\alpha+r_1+r_2+1)(r_3+r_4)} \cdot \mathrm{Wr}\left[f_1,\ldots,f_r,h\right](x,\lambda),$$

$$\psi^{\alpha}_{M_1,M_2}(x,\lambda) := e^{-(r_2+r_4)x} x^{(\alpha+r_1+r_2)(r_3+r_4+1)} \cdot \mathrm{Wr}\left[f_1,\ldots,f_r,\widetilde{h}\right](x,\lambda).$$

Corollary

Let M_1 and M_2 be given as above.

(a) If $n_{r_1} = 0$, then for some $d \in \mathbb{N}_0$

$$\phi^{\alpha}_{M_1,M_2}(x,\lambda) = \left(\frac{-\lambda}{\alpha+1} \prod_{i=1}^{r_3} (m_i' - \alpha) \prod_{i=1}^{r_4} (n_i' + 1) \right) \phi^{\alpha+1}_{M_1-1,M_2}(x,\lambda-1).$$

(b) If $n'_{r_4} = 0$, then for some $d \in \mathbb{N}_0$

$$\phi^{\alpha}_{M_1,M_2}(x,\lambda) = (-1)^d \left(\alpha \prod_{i=1}^{r_1} (n_i + 1) \prod_{i=1}^{r_2} (m_i + \alpha) \right) \phi^{\alpha-1}_{M_1+1,M_2}(x,\lambda+1).$$

Corollary (cont.)

Let M_1 and M_2 be given as above.

(c) If $m_{r_2} = 0$, then for some $d \in \mathbb{N}_0$

$$\phi^{\alpha}_{M_1,M_2}(x,\lambda) = (-1)^d \left(\frac{\alpha+1+\lambda}{-\lambda} \prod_{i=1}^{r_3}(m_i'+1) \prod_{i=1}^{r_4}(n_i'-\alpha) \right) \phi^{\alpha+1}_{M_1,M_2-1}(x,\lambda).$$

(d) If $m'_{r_3} = 0$, then for some $d \in \mathbb{N}_0$

$$\phi^{\alpha}_{M_1,M_2}(x,\lambda) = (-1)^d \left(\alpha \prod_{i=1}^{r_1}(n_i+\alpha) \prod_{i=1}^{r_2}(m_i+1) \right) \phi^{\alpha-1}_{M_1,M_2+1}(x,\lambda).$$

Theorem 1

Let $f_1, \ldots, f_r$, M_1 and M_2 be as defined above, and μ and ν be their associated partitions after shifts t_1 and t_2, respectively. Then for $\lambda \neq n_i$ with $i = 1, \ldots, r_1$,

$$\Phi^{\alpha}_{M_1,M_2}(x, \lambda) = C_1(\alpha, \lambda, M_1, M_2) C_2(\alpha, M_1, M_2) \phi^{\alpha - t_1 - t_2}_{\mu,\nu}(x, \lambda + t_1)$$

where

$$C_2(\alpha, \lambda, M_1, M_2) = (-1)^d \prod_{j=1}^{r_1} \prod_{k=1}^{r_3} \left(m'_k - \alpha - n_j\right) \prod_{j=1}^{r_2} \prod_{k=1}^{r_4} \left(n'_k - \alpha - m_j\right) \times \prod_{j=1}^{r_1} \prod_{k=1}^{r_4} \left(n_j + n'_k + 1\right) \prod_{j=1}^{r_2} \prod_{k=1}^{r_3} \left(m_j + m'_k + 1\right),$$

for some $d \in \mathbb{N}_0$.

Theorem 1 (cont.)

The constant $C_1(\alpha, \lambda, M_1, M_2)$ depends on whether t_1 and t_2 are positive or negative:

$$C_1(\alpha,\lambda,M_1,M_2) = \frac{(-\lambda)^{(|t_1|)}(\alpha+1+\lambda)^{(|t_2|)}}{(\alpha+1)^{(|t_1|)}\lambda^{|t_2|}}, \qquad \text{for } t_1, t_2 < 0,$$

$$C_1(\alpha,\lambda,M_1,M_2) = \frac{(\alpha)_{(t_1)}(\alpha+1+\lambda)^{(|t_2|)}}{\lambda^{|t_2|}\cdot\prod_k(-\lambda-k-1)}, \qquad \text{for } t_1 > 0,\ t_2 < 0,$$

$$C_1(\alpha,\lambda,M_1,M_2) = \frac{(-\lambda)^{(|t_1|)}(-\lambda)^{t_2-r_3}\cdot\prod_{i=1}^{r_3}(\alpha-m_i')}{(\alpha+1)^{(|t_1|)}\prod_{k'}(\lambda+\alpha-k')}, \qquad \text{for } t_1 < 0,\ t_2 > 0,$$

$$C_1(\alpha,\lambda,M_1,M_2) = \frac{(\alpha)_{(t_1)}(-\lambda)^{t_2-r_3}\cdot\prod_{i=1}^{r_3}(\alpha-m_i')}{\prod_k(-\lambda-k-1)\prod_{k'}(\lambda+\alpha-k')}, \qquad \text{for } t_1, t_2 > 0,$$

where $k \in \{0, \ldots, t_1 - 1\}$ such that $k \notin \{n_j'\}_{j=1}^{r_4}$ and $k' \in \{0, \ldots, t_2 - 1\}$ such that $k' \notin \{m_j'\}_{j=1}^{r_3}$. If either t_1 or t_2 are 0, we use the convention that the above formulas hold without the contributions from that Maya diagram.

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