Multi-scale analysis for the random XXZ higher spin chain

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Context for the XXZ Spin-J Project

- Previous results:
  - Beaud and Warzel (2017), and Elgart, Klein, and Stolz (2017) proved localization in the droplet spectrum (bottom of the spectrum) for the XXZ spin-1/2 chain via the fractional moment method. Elgart, Klein, and Stolz also proved dynamical exponential clustering of the averaged correlations of local observables and some proper spin chain manifestations of localization.
  - Recently Klein and Elgart developed a multiscale analysis from which they derive localization for the XXZ spin-1/2 chain.
  - Fischbacher and Ogunkoya (2020) classified the minimal configurations and derived entanglement bounds for the XXZ spin-J chain.

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  - Recently Klein and Elgart developed a multiscale analysis from which they derive localization for the XXZ spin-1/2 chain.
  - Fischbacher and Ogunkoya (2020) classified the minimal configurations and derived entanglement bounds for the XXZ spin-J chain.
- This presentation:
  - Localization in the droplet spectrum of the Spin-J quantum spin chain.

The XXZ Model

$$H_\omega = \sum_{i\in\mathbb{Z}} \left(J^2 - \frac{1}{\Delta}(S_i^1 S_{i+1}^1 + S_i^2 S_{i+1}^2) - S_i^3 S_{i+1}^3 \right) + \lambda \sum_{i\in\mathbb{Z}} \omega_i \mathcal{N}_i$$

- Formal operator on $\bigotimes_{i\in\mathbb{Z}} \mathbb{C}^{2J+1}$.
- A spin number $J \in \frac{1}{2}\mathbb{N} = \{\frac{1}{2}, 1, \frac{3}{2}, \dots\}$
- $\Delta > 2J$ specifies the Ising phase of the XXZ - chain.
- $\lambda > 0$ is the disorder parameter.
- Let $\{e_i\}_{i=0}^{2J}$ be the canonical basis for $\mathbb{C}^{2J+1}$

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Pauli Spin-J Matrices

$$S^- e_i = \begin{cases} \sqrt{2J + i(2J-1) - i^2}\, e_{i+1} & \text{if } i < 2J \\ 0 & \text{if } i = 2J \end{cases}$$

$$S^+ e_i = \begin{cases} \sqrt{i(2J+1) - i^2}\, e_{i-1} & \text{if } i > 0 \\ 0 & \text{if } i = 0 \end{cases}$$

$$S^1 = \tfrac{1}{2}(S^+ + S^-), \quad S^2 = \tfrac{1}{2i}(S^+ - S^-)$$

$$S^3 = \mathrm{diag}(J, J-1, \ldots, -J+1, -J)$$

The XXZ Model

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- $\mathcal{N}$ is the particle number operator.

$$\mathcal{N} = 2J - S^3 = \mathrm{diag}(0, 1, 2, \ldots, 2J).$$

- $\omega = \{\omega_i\}_{i\in\mathbb{Z}}$ are i.i.d random variables with probability distribution μ absolutely continuous, bounded density, and $\{0,1\} \subset \mathrm{supp}\mu \subset [0,1]$.
- H_ω is self-adjoint on an appropriate Hilbert space $\mathcal{H}$ constructed from $\bigotimes_{\mathbb{Z}} \mathbb{C}^{2J+1}$.

The XXZ Model

$$H_\omega = \sum_{i\in\mathbb{Z}} \left(J^2 - \frac{1}{\Delta}(S_i^1 S_{i+1}^1 + S_i^2 S_{i+1}^2) - S_i^3 S_{i+1}^3 \right) + \lambda \sum_{i\in\mathbb{Z}} \omega_i \mathcal{N}_i$$

- If $\Lambda \subset \mathbb{Z}$ (finite), then H_Λ is appropriately defined on $\bigotimes_\Lambda \mathbb{C}^{2J+1}$
- Total particle number operator

$$\mathcal{N}_\Lambda = \sum_{i\in\Lambda} \mathcal{N}_i.$$

- We have that $[\mathcal{N}_\Lambda, H_\Lambda] = 0$, particle numbers are conserved by H_Λ.
- Motivates $H_\Lambda^{(N)}$, the restriction of H_Λ to an N-particle subspace.

Projection Operators

Definition: Projections

$$P_\Lambda^+ := \bigotimes_{x\in\Lambda} \pi_{e_0}(x),$$
$$P_\Lambda^- := 1 - P_\Lambda^+.$$

- Here $\pi_{e_0}(x)$ is the orthogonal projection onto $\ker(\mathcal{N}_x)$.
- P_Λ^+ is the orthogonal projection onto the state where no particles are present in Λ (vacuum).
- Conversely P_Λ^- is the projection onto the space of configurations with at least one particle in Λ.

Target Theorem and Regularity

Definition: (m,E)-regular

Given $E \in \mathbb{R}$ and $m > 0$, an interval $\Lambda_L(j)$ is said to be (m, E)-regular if

$$m > L^{-\kappa}, \quad \text{dist}(E, \sigma(H_{\Lambda_L(j)})) > e^{-L^\beta}$$

$$\|P_i^-(H_{\Lambda_L(j)} - E)^{-1} P^+_{\Lambda_R(i)\cap\Lambda_L(j)}\| \leq e^{-m(R+1)} \text{ for all } i \in \Lambda_L(j) \text{ and } R > L^\tau.$$

The equation in red is not the Green's function! This is the appropriate substitute for this model.

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$$\mathcal{R}(m, L, I, u, v) := \{E \in I \implies \Lambda_L(u) \text{ or } \Lambda_L(v) \text{ is } (m, E) - \text{regular.}\}$$

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The equation in red is not the Green's function! This is the appropriate substitute for this model.

Theorem: The Multiscale Analysis

Fix $0 < \zeta < 1$, let $\Delta > 2J$, $\lambda > 0$, and $\delta \in (0, 1)$ and suppose that Δ and λ are large enough. Then there is $\mathcal{L} = \mathcal{L}(\Delta, \lambda, \delta)$ and $m = m(\Delta, \lambda, \delta)$ such that for all $L \geq \mathcal{L}$ and $u, v \in \mathbb{Z}$ with $|u - v| > 2L$ we have

$$\mathbb{P}\{\mathcal{R}(m, L, I_{1,\delta}, u, v)\} \geq 1 - e^{-L^\xi}.$$

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$$\mathbb{P}\{\mathcal{R}(m, L, I_{1,\delta}, u, v)\} \geq 1 - e^{-L^{\xi}}.$$

- This theorem implies eigenfunction localization.

Eigenfunction Localization

Definition: (m,I)-localizing

For $I \subset I_{1,\delta}$ and $m > 0$ we will say that $\Lambda_L \subset \mathbb{Z}$ is (m, I)-localizing if an eigensystem $\{(\varphi_\nu, \nu)\}_{\nu \in \sigma(H_{\Lambda_L})}$ is (m, I)-localized, that is, for all $\nu \in \sigma(H_{\Lambda_L}) \cap I$ there is $j_\nu \in \Lambda_L$ such that φ_ν is (j_ν, m)-localized:

$$\|P_i^- \varphi_\nu\| \leq e^{-m|i-j_\nu|} \text{ for all } i \in \Lambda_L \text{ with } |i - j_\nu| \geq L^\tau.$$

Event: $\mathcal{Q}(m, L, I, u) = \{\Lambda_L(u) \text{ is } (m, I) - \text{localizing for } H\}.$

Eigenfunction Localization

Theorem: Eigenfunction Localization

Fix $0 < \xi < 1$, let $\Delta > 2J$, $\lambda > 0$, and $\delta \in (0,1)$ and suppose that Δ and λ are large enough. Then there is $\mathcal{L} = \mathcal{L}(\Delta, \lambda, \delta)$ and $m = m(\Delta, \lambda, \delta)$ such that for all $L \geq \mathcal{L}$ and $u \in \mathbb{Z}$ we have

$$\mathbb{P}\{\mathcal{Q}(m, L, I_{1,\delta}, u)\} \geq 1 - e^{-L^{\xi}}.$$

Moreover if $\omega \in \mathcal{Q}(m, L, I_{1,\delta}, u)$ and $\{\varphi_\nu, \nu\}_{\nu \in \sigma(H_{\Lambda_L(u)})}$ is an eigensystem for $H_{\Lambda_L(u)}$, then for all $i, j \in \Lambda_L(u)$ with $|i - j| \geq L^{\tilde{\tau}}$ ($\tau < \tilde{\tau} < 1$),

$$\sum_{\nu \in \sigma(H_{\Lambda_L(u)}) \cap I_{1,\delta}} \|P_i^- \varphi_\nu\| \|P_j^- \varphi_\nu\| \leq e^{-\frac{m}{2}|i-j|}.$$

Unitary equivalence to a Schrödinger Operator

$$U_\Lambda^{(N)} : (\bigotimes_{i\in\Lambda} \mathbb{C}_i^{2J+1})^{(N)} \to \ell^2(\mathbf{M}_\Lambda^{(N)}) \text{ unitary.}$$

$$U_\Lambda^{(N)} H_\Lambda^{(N)} \left(U_\Lambda^{(N)}\right)^* = -\frac{1}{2\Delta} A_\Lambda^{(N)} + \mathcal{W}_\Lambda^{(N)} + \lambda V_{\Lambda,\omega}^{(N)} =: \widehat{H}_\Lambda^{(N)}$$

The Space of Configurations

$$\mathbf{M}_\Lambda^{(N)} := \left\{ \mathbf{m} : \Lambda \to \{0, 1, \ldots, 2J\} : \sum_{x\in\Lambda} \mathbf{m}(x) = N \right\} ,$$

Configuration adjacency, $\mathbf{m} \sim \mathbf{n}$:

$$\exists \{x_0, x_1\} \in \mathcal{E}_\Lambda \text{ such that } \mathbf{m}(x_0) = \mathbf{n}(x_0) + 1,$$
$$\mathbf{m}(x_1) = \mathbf{n}(x_1) - 1,$$
$$\text{and } \mathbf{m}(x) = \mathbf{n}(x) \text{ when } x \in \Lambda \setminus \{x_0, x_1\}.$$

Unitary equivalence to a Schrödinger Operator

$$U_\Lambda^{(N)} H_\Lambda^{(N)} \left(U_\Lambda^{(N)}\right)^* = -\frac{1}{2\Delta} A_\Lambda^{(N)} + \mathcal{W}_\Lambda^{(N)} + \lambda V_{\Lambda,\omega}^{(N)} =: \widehat{H}_\Lambda^{(N)}$$

Weighted Adjacency Operator

$$(A_\Lambda^{(N)} f)(\mathbf{m}) = \sum_{\mathbf{n}:\mathbf{n}\sim\mathbf{m}} w(\mathbf{m},\mathbf{n}) f(\mathbf{n})$$

$$w(\mathbf{m},\mathbf{n}) = \prod_{x:\mathbf{m}(x)\neq\mathbf{n}(x)} \left(J(\mathbf{m}(x)+\mathbf{n}(x)+1) - \mathbf{m}(x)\mathbf{n}(x)\right)^{1/2}$$

Unitary equivalence to a Schrödinger Operator

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The $\mathcal{W}$ function

$$(\mathcal{W}_\Lambda^{(N)} f)(\mathbf{m}) = \mathcal{W}_\Lambda^{(N)}(\mathbf{m}) f(\mathbf{m}) = \left(2JN - \sum_{\{i,i+1\}\in\mathcal{E}(\Lambda)} \mathbf{m}(i)\mathbf{m}(i+1)\right) f(\mathbf{m})$$

The Random Potential

$$(V_{\Lambda,\omega}^{(N)} f)(\mathbf{m}) = V_{\Lambda,\omega}(\mathbf{m}) f(\mathbf{m}) = \left(\sum_{x\in\Lambda} \mathbf{m}(x)\omega_x\right) f(\mathbf{m})$$

Unitary equivalence to a Schrödinger Operator

$$U_\Lambda^{(N)} H_\Lambda^{(N)} \left(U_\Lambda^{(N)}\right)^* = -\frac{1}{2\Delta} A_\Lambda^{(N)} + \mathcal{W}_\Lambda^{(N)} + \lambda V_{\Lambda,\omega}^{(N)} =: \widehat{H}_\Lambda^{(N)}$$

$$\begin{aligned} Q_1 &:= 4J^2 + 2J. \\ \mathbf{M}_{\Lambda,1}^{(N)} &:= \left\{\mathbf{m} \in \mathbf{M}_\Lambda^{(N)} : \mathcal{W}_\Lambda^{(N)}(\mathbf{m}) < Q_1\right\} \\ P_{\Lambda,1}^{(N)} &\text{ is the orthogonal projection onto } \ell^2(\mathbf{M}_{\Lambda,1}^{(N)}) \\ H_0^{(N)} &\geq \left(1 - \tfrac{2J}{\Delta}\right) \mathcal{W}^{(N)} \\ I_1 &:= \left[4J^2 \left(1 - \tfrac{2J}{\Delta}\right), Q_1 \left(1 - \tfrac{2J}{\Delta}\right)\right) \\ I_{1,\delta} &:= \left[4J^2 \left(1 - \tfrac{2J}{\Delta}\right), (Q_1 - \delta) \left(1 - \tfrac{2J}{\Delta}\right)\right] \end{aligned}$$

- Configurations in $\mathbf{M}_{\Lambda,1}^{(N)}$ have support with one connected component.
- I_1 is called the droplet spectrum.

Large Deviation Estimates

Definition: Λ_ℓ is $(1,\mathcal{N})$-reduced

$$\lambda V_\omega P^{(N)}_{\Lambda_\ell,1} \geq Q_1 \left(1 - \frac{2J}{\Delta}\right) P^{(N)}_{\Lambda_\ell,1} \text{ for all } N > \ell^{\zeta'}. \quad (0 < \zeta' < \zeta < 1)$$

$$\{\Lambda_\ell \text{ is } (1,\mathcal{N})\text{-reduced}\} \subset \{I_1 \cap \sigma(H^{(N)}_\Lambda) = \emptyset \text{ for all } N > \ell^{\zeta'}\}.$$

Theorem: $(1,\mathcal{N})$-reduced probability estimate

$$\mathbb{P}\{\Lambda_\ell \text{ is } (1,\mathcal{N})\text{-reduced}\} \geq 1 - e^{-c_\mu \ell^{\zeta'}}.$$

- If Λ_ℓ is $(1,\mathcal{N})$-reduced then for localization we only worry about $N \leq \ell^{\zeta'}$.

Lemma: A Deterministic Estimate.

Let $\Lambda = \Lambda_L(i)$, $E \in I_{1,\delta} \setminus \sigma(H_\Lambda^{(N)})$, $\Theta \subset \mathbf{M}_\Lambda^{(N)}$, $0 \le q \le \ell \le L$, $\mathbf{S}_{\Lambda_q(i)} \cap \Theta \subset (\mathbf{M}_{\Lambda,1}^{(N)})^c$, and

$$q < \tilde{\ell} \le \mathrm{dist}_\Lambda(\mathbf{S}_{\Lambda_q(i)} \cap \Theta, \mathbf{M}_{\Lambda,1}^{(N)}) + q.$$

$$m = \log\left(1 + \frac{\delta(\Delta - 2J)}{4JQ_1}\right).$$

Then for all $\Psi \subset \mathbf{M}_\Lambda^{(N)}$ we have

$$\left\| P_{\Lambda_q(i)}^{-} \chi_\Theta (H_\Lambda^{(N)} - E)^{-1} \chi_\Psi \right\| \le \frac{C_1(J,\Delta)}{\delta} e^{-m \cdot \mathrm{dist}_\Lambda(\mathbf{S}_{\Lambda_q(i)} \cap \Theta, \Psi)}$$
$$+ \frac{C_2(J,\Delta)}{\delta} \sum_{r \in \Lambda} e^{-m\left(1 - \frac{q}{\tilde{\ell}}\right)\max\{|r-i|, \tilde{\ell}\}} \left\| P_r^{-} \left(H_\Lambda^{(N)} - E\right)^{-1} \chi_\Psi \right\|$$

Regular Intervals

Lemma: Regular Intervals Estimate

Let $E \in I_{1,\delta}$. Assume that the interval Λ_L is $(1, \mathcal{N})$-reduced. Let m satisfy

$$\ell^{-\kappa} < m \leq \log\left(1 + \frac{\delta(\Delta - 2J)}{4JQ_1}\right).$$

Let $i, j \in \Lambda_L$ with $|j - i| < R - 2\ell$ so that $\Lambda_\ell(i) \subset \Lambda_R(j)$, and suppose that the interval $\Lambda_\ell(i)$ is (m, E)-regular. Then for sufficiently large L,

$$\|P_i^- R_{\Lambda_L}(E) P_{\Lambda_R(j)}^+\| \leq \max\left\{ e^{-m'(R+1-|j-i|)}, \max_{r \in \Lambda_L} e^{-m' \max\{|r-i|, \ell^\tau\}} \left\| P_r^- R_{\Lambda_L}(E) P_{\Lambda_R(j)}^+ \right\| \right\}$$

where $m' \geq m(1 - C\ell^{-(\tau - \beta - \kappa)})$.

Buffered Sets (optional)

Definition: (m,E)-Buffer

Interval $\Upsilon \subset \Lambda_L$ is called an (m,E)-buffer if for all $s \in \partial^{\Lambda_L}\Upsilon$ we have that $\Lambda_\ell(s)$ is an (m,E)-regular interval. In this case we set $\Upsilon' = \Upsilon \setminus \partial^{\Lambda_L}\Upsilon$.

Lemma: Buffered Set Estimate

Let $E \in I_{1,\delta}$, Λ_L is $(1,\mathcal{N})$-reduced, $\Upsilon \subset \Lambda_L$ be an (m,E)-buffer, where m satisfies the inequality, $j \in \Lambda_L$ and $\Upsilon \subset \Lambda_R(j)$. Assume $\text{dist}(E, \sigma(H_{\Lambda_L \setminus \partial\Upsilon})) > e^{-L^\beta}$. Then there exist $s_\Upsilon \in \partial_{\Lambda_L}\Upsilon$ such that for all $q \in \Upsilon'$ we have

$$\|P_q^- R_L(E) P_{\Lambda_R(j)}^+\|$$

$$\lesssim e^{L^\beta} \max\left\{ e^{-m'(R+1-|s_\Upsilon - j|)}, \max_{r\in\Lambda_L} e^{-m' \max\{|r-s_\Upsilon|, \ell^\tau\}} \|P_r^- R_L(E) P_{\Lambda_R(j)}^+\| \right\}.$$

The Starting Condition

Theorem: Starting Condition

Given $\Delta > 2J$, $\lambda > 0$ and $\delta \in (0,1)$. Suppose that L satisfies,

$$\max\left\{\frac{4JQ_1}{\bar{\mu}}\left(1-\tfrac{2J}{\Delta}\right)L^{-\zeta'}, e^{-\frac{1}{6}L^{\beta}}\right\} \leq \lambda$$

$$e^{-L^{\beta}} < \frac{\delta}{2}\left(1-\tfrac{2J}{\Delta}\right)$$

$$L^{-\kappa} < \frac{1}{3}\log\left(1+\frac{\delta(\Delta-2J)}{4JQ_1}\right)$$

$$e^{L^{\zeta''}} \leq \Delta\lambda$$

Let $m = \frac{1}{4}\min\left\{1, \log\left(1+\frac{\delta(\Delta-2J)}{4JQ_1}\right)\right\}$. Then if L is sufficiently large, setting $\theta_L = \min\{2mL^{\tau}e^{-4mL-L^{\beta}}, e^{-L^{\beta}}\}$, for all $E \in I_{1,\delta}$ we have

$$\mathbb{P}\{\mathcal{R}(m, L, I(E,\theta_L), u, v)\} \geq 1 - e^{-L^{\zeta}} \text{ for all } u, v \in \mathbb{Z} \text{ with } |u-v| > 2L.$$

The Multiscale Analysis

Theorem: Induction on Scales

Given $\delta \in (0,1)$ and $0 < \zeta < 1$, let $\Delta > 2J$ and $\lambda > 0$, the scale L_0, and m_0 satisfy the hypothesis of the starting condition. Consider an interval $I \subset I_{1,\delta}$, and suppose we have

$$\mathbb{P}\{\mathcal{R}(m_0, L_0, I, u, v)\} \geq 1 - e^{-L_0^{\zeta}} \text{ for all } u, v \in \mathbb{Z} \text{ with } |u - v| > 2L_0.$$

Then, if L_0 is sufficiently large, setting $L_{k+1} = L_k^{\gamma}$, we have

$$\mathbb{P}\{\mathcal{R}(m_k, L_k, I, u, v)\} \geq 1 - e^{-L_k^{\zeta}} \text{ for all } u, v \in \mathbb{Z} \text{ with } |u - v| > 2L_k,$$

for all $k = 0, 1, \ldots$ Also m_k is a decreasing sequence with $m_k \geq m_0/2$.

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for all $k = 0, 1, \ldots$ Also m_k is a decreasing sequence with $m_k \geq m_0/2$.

- Proof by first estimating the size/amount of non-regular intervals,
- then iterating the results for buffer-sets and regular intervals to prove regularity for one of the larger intervals.

Thank you.

Extra Material

Finite XXZ-Spin Systems

Spin Systems

Let $\Lambda = (\mathcal{V}, \mathcal{E})$ be a finite graph. We will consider operators on the tensor product

$$\mathcal{H}_\Lambda = \bigotimes_{v \in \Lambda} \mathbb{C}^{2J+1}.$$

XXZ two site Hamiltonian

Let $u, v \in \mathcal{V}$ with $u \sim v$ then for $\Delta > 2J > 0$,

$$h_{u,v} = J^2 - \frac{1}{\Delta}(S_u^1 S_v^1 + S_u^2 S_v^2) - S_u^3 S_v^3$$

Particle Number

$$\mathcal{N} = 2J - S^3 = \mathrm{diag}(0, 1, 2, \ldots, 2J).$$

The XXZ Hamiltonian

Adjusted two site Hamiltonian

$$\tilde{h}_{u,v} = h_{u,v} - J(\mathcal{N}_u + \mathcal{N}_v) = -\mathcal{N}_u\mathcal{N}_v - \frac{1}{2\Delta}(S_u^+ S_v^- + S_u^- S_v^+)$$

The Full Hamiltonian

$$H_\Lambda = \tilde{H}_\Lambda + 2J\mathcal{N}_\Lambda + \lambda V_{\Lambda,\omega}$$

$$\tilde{H}_\Lambda = \sum_{u,v\in\mathcal{E}} \tilde{h}_{u,v}, \quad \mathcal{N}_\Lambda = \sum_{u\in\mathcal{V}} \mathcal{N}_u, \quad V_{\Lambda,\omega} = \sum_{u\in\mathcal{V}} \omega_u \mathcal{N}_u$$

Conservation and Decomposition

$$[\mathcal{N}_G, H_G] = 0 \implies \mathcal{H}_\Lambda = \bigoplus_{N=0}^{2J\#(\Lambda)} \mathcal{H}_\Lambda^{(N)}$$

Interesting Properties 1. Bounds and Minimizers

Lemma: Relative Bounds

$$-4J\mathcal{W}_\Lambda \le A_\Lambda \le 4J\mathcal{W}_\Lambda$$
$$\left(1 - \tfrac{2J}{\Delta}\right)\mathcal{W}_\Lambda \le H_\Lambda \le \left(1 + \tfrac{2J}{\Delta}\right)\mathcal{W}_\Lambda.$$

Cite Christoph.

Lemma: Minimizers of $\mathcal{W}$.

Let Λ be a finite interval and let $N \in \mathbb{N}$.

$$\mathcal{W}_0^{(N)} = \begin{cases} 2JN - \lfloor \frac{N}{2} \rfloor \lceil \frac{N}{2} \rceil & N < 4J \\ 4J^2 & N \ge 4J \end{cases}$$

Proof cite Christoph and myself. Structure minimal configurations are known but not important for this talk.

Interesting Properties 2. Energy Intervals

$$\mathbf{M}_{\Lambda,k}^{(N)} := \left\{ \mathbf{m} \in \mathbf{M}_{\Lambda}^{(N)} : \mathcal{W}_{\Lambda}^{(N)}(\mathbf{m}) < 4J^2 + 2Jk \right\}$$

$$Q_k = 4J^2 + 2Jk.$$

$$I_k := \left[4J^2\left(1 - \tfrac{2J}{\Delta}\right), Q_k\left(1 - \tfrac{2J}{\Delta}\right)\right)$$

$$I_{k,\delta} := \left[4J^2\left(1 - \tfrac{2J}{\Delta}\right), (Q_k - \delta)\left(1 - \tfrac{2J}{\Delta}\right)\right]$$

$P_{\Lambda,k}^{(N)}$ is the orthogonal projection onto $\ell^2(\mathbf{M}_{\Lambda,k}^{(N)})$

Conjecture: Clusters and $\mathcal{W}$

For all $N \geq 4kJ$, $\mathbf{m} \in \mathbf{M}_{\Lambda,k}^{(N)}$ if and only if $\mathbf{m}$ is a configuration with at most k connected components. The case for $k = 1$ is known and proven in [??].

Interesting Properties 3. Lifting the Spectrum

Definition: The Lifted Operator

$$H_{\Lambda,k}^{(N)} := H_{\Lambda}^{(N)} + (Q_k - 1)\left(1 - \tfrac{2J}{\Delta}\right) P_{\Lambda,k}^{(N)}.$$

Lemma: Lifting the Spectrum ($N > 0$)

$$H_{\Lambda,k}^{(N)} \geq Q_k \left(1 - \tfrac{2J}{\Delta}\right).$$

Proof. From Lemma: Relative Bounds

$$\begin{aligned}
\frac{H_{\Lambda,k}^{(N)}}{\left(1 - \frac{2J}{\Delta}\right)} &\geq \mathcal{W} + (Q_k - 1)P_{\Lambda,k}^{(N)} \\
&= \left(\mathcal{W} + (Q_k - 1)P_{\Lambda,k}^{(N)}\right) P_{\Lambda,k}^{(N)} + \left(\mathcal{W} + (Q_k - 1)P_{\Lambda,k}^{(N)}\right) \overline{P}_{\Lambda,k}^{(N)} \\
&\geq Q_k P_{\Lambda,k}^{(N)} + \mathcal{W}\overline{P}_{\Lambda,k}^{(N)} \geq Q_k.
\end{aligned}$$

Equivalence to a Schrödinger Operator 1. Adjacency Op

The Equivalence

$$U_\Lambda^{(N)} H_\Lambda^{(N)} \left(U_\Lambda^{(N)} \right)^* = -\frac{1}{2\Delta} A_\Lambda^{(N)} + \mathcal{W}_\Lambda^{(N)} + \lambda V_{\Lambda,\omega}^{(N)} =: \widehat{H}_\Lambda^{(N)}$$

Weighted Adjacency Operator

$$(A_\Lambda^{(N)} f)(\mathbf{m}) = \sum_{\mathbf{n}:\mathbf{n}\sim\mathbf{m}} w(\mathbf{m},\mathbf{n}) f(\mathbf{n})$$

$$w(\mathbf{m},\mathbf{n}) = \prod_{x:\mathbf{m}(x)\neq\mathbf{n}(x)} \left(J(\mathbf{m}(x)+\mathbf{n}(x)+1) - \mathbf{m}(x)\mathbf{n}(x) \right)^{1/2}$$

Equivalence to a Schrödinger Operator 2. The Potentials

$$\widehat{H}_\Lambda^{(N)} = -\frac{1}{2\Delta} A_\Lambda^{(N)} + \mathcal{W}_\Lambda^{(N)} + \lambda V_{\Lambda,\omega}^{(N)}$$

The $\mathcal{W}$ function

$$(\mathcal{W}_\Lambda^{(N)} f)(\mathbf{m}) = \mathcal{W}_\Lambda^{(N)}(\mathbf{m}) f(\mathbf{m}) = \left(2JN - \sum_{\{i,i+1\}\in\mathcal{E}(\Lambda)} \mathbf{m}(i)\mathbf{m}(i+1) \right) f(\mathbf{m})$$

The Random Potential

$$(V_{\Lambda,\omega}^{(N)} f)(\mathbf{m}) = V_{\Lambda,\omega}(\mathbf{m}) f(\mathbf{m}) = \left(\sum_{x\in\Lambda} \mathbf{m}(x)\omega_x \right) f(\mathbf{m})$$

ω_x i.i.d Hölder cont, $\sup_{a\in\mathbb{R}} \mu\{[a, a+t]\} \leq K t^\alpha$ for all $t \in [0, 1]$.

Step 1. Large Deviation 1.

Definition: Λ_ℓ is $(1,\mathcal{N})$-reduced

if for all $N > \ell^{\zeta'}$,
$$\lambda V_\omega P_{\Lambda,1}^{(N)} \geq Q_1 \left(1 - \tfrac{2J}{\Delta}\right) P_{\Lambda,1}^{(N)}.$$

$$\begin{aligned} H_\Lambda^{(N)} &\geq \left(1 - \tfrac{2J}{\Delta}\right) \mathcal{W} + \lambda V_\omega \\ &= \left[\left(1 - \tfrac{2J}{\Delta}\right) \mathcal{W} + \lambda V_\omega\right] P_{\Lambda,1}^{(N)} + \left[\left(1 - \tfrac{2J}{\Delta}\right) \mathcal{W} + \lambda V_\omega\right] \bar{P}_{\Lambda,1}^{(N)} \\ &\geq \left(1 - \tfrac{2J}{\Delta}\right) \left(4J^2 + Q_1\right) P_{\Lambda,1}^{(N)} + \left(1 - \tfrac{2J}{\Delta}\right) Q_1 \bar{P}_{\Lambda,1}^{(N)} \geq Q_1 \left(1 - \tfrac{2J}{\Delta}\right). \end{aligned}$$

$$\{\Lambda_\ell \text{ is } (1,\mathcal{N})\text{-reduced}\} \subset \{I_1 \cap \sigma(H_\Lambda^{(N)}) = \emptyset \text{ for all } N > \ell^{\zeta'}.\}$$

Theorem: $(1,\mathcal{N})$-reduced probability estimate

$$\mathbb{P}\{\Lambda_\ell \text{ is } (1,\mathcal{N})\text{-reduced}\} \geq 1 - e^{-c_\mu \ell^{\zeta'}}.$$

Target Theorem Proof Strategy

1. (m, E)-regular requires $\text{dist}(E, \sigma(H_{\Lambda_L(j)})) > e^{-L^\beta}$, need to control the probability.
   - Reducedness and Wegner Estimates.
2. Need to be able to move between scales. If a regular box sits inside of larger box; what can be said about the "Greens function" on the larger box?
   - Combes-Thomas estimates and initial localization lemmas.
3. What about intervals which are not regular?
   - Need to estimate the probability of this.
   - Need to estimate the size of the resolvent here too.
4. Does the "large enough" scale L_0 actually exist?
   - The starting condition.
5. If an interval of size L_0^γ contains a large amount of regular intervals of size L_0, then the larger interval is also regular.
6. Induct on L_0, ie $L_0, L_0^\gamma, L_0^{\gamma^2}, \ldots$ then move to arbitrary scales.

Step 1. Large Deviation 2.

Lemma: Wegner Estimate

Let I be an open interval such that $I \subset I_1$. Then

$$\mathbb{P}\{\sigma_I(H_\Lambda^{(N)}) \neq \emptyset\} \leq K|I|^\alpha \lambda^{-\alpha} \ell^{2Q_1+1}$$

Notice in particular,

$$\begin{aligned}\mathbb{P}\left\{\sigma_I(H_\Lambda) \neq \emptyset\right\} &\leq \mathbb{P}\left\{\sigma_I(H_\Lambda^{(N)}) \neq \emptyset \text{ for some } N \leq \ell^{\zeta'}\right\} \\ &+ \mathbb{P}\left\{\Lambda \text{ is not } (1, \mathcal{N}) - \text{reduced}\right\} \\ &\leq K|I|^\alpha \lambda^{-\alpha} \ell^{2Q_1+1} + e^{-c_\mu \ell^{\zeta'}}.\end{aligned}$$

Suppose $E \in I_{1,\delta}$ and $I = (E - e^{-\ell^\beta}, E + e^{-\ell^\beta})$. Finishes Step 1.

Step 5. The Multiscale Analysis 2. Proof Sketch part 1.

- Let $S_\ell = 2\lfloor \ell^{(\gamma-1)\zeta_*} \rfloor$.
- Careful reasoning and comparing with the $\mathcal{R}(\dots)$ event in the hypothesis can give an estimate

$$\mathbb{P}\{\Lambda_L \text{ has at least } S_\ell \text{ nonregular disjoint subintervals}\} \leq e^{-L^\zeta}$$

- Estimate the size of the buffer, Υ, required for ω in the complimentary event.

$$|\Upsilon| \leq 6\ell(S_\ell + 1) \leq 12\ell^{(\gamma-1)\zeta_*-1} < L^\tau.$$

- Use Wegner and large deviation estimates to control the probability that $\operatorname{dist}(E, \sigma(H'_{\Lambda_L,K})) > e^{-L^\beta}$ in for all $K \in \mathcal{K}$, a large collection of subintervals.
- Pick ω so that Λ_L is $(1, \mathcal{N})$ reduced, this occurs with high probability.

Step 5. The Multiscale Analysis 3. Proof Sketch part 2.

- Once ω is chosen in the high probability set we can iterate the localization and buffered subsets lemmas.
- Let
$$G(r) = \|P_r^-(H_L - E)^{-1} P^+_{\Lambda_R(i)}\| \text{ for } r \in \Lambda_L.$$
- The Localization lemma,
$$G(j) \leq \max\left\{ e^{-m^{(1)}(R+1-|j-i|)}, \max_{r\in\Lambda_L} e^{-m^{(1)}\max\{|r-j|,\ell^{\tau}\}} G(r) \right\}.$$
- We can iterate the previous equation to get,
$$G(i) \leq e^{-m^{(2)}(|r_* - i| - 2\ell^{\gamma_*})} e^{L^{\beta}} \leq e^{-m^{(2)}(R-4\ell^{\gamma_*})} e^{L^{\beta}} \leq e^{-m^{(3)}(R+1)}.$$
- The desired bound for $(m^{(3)}, E)$-regularity.