Multi-orthogonality and bulk queueing theory

Baylor Analysis Fest congratulating Professor Littlejohn

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Model of queues with a service of m admitted customers

Kendall's notation: $M/M(m,m)/1$. Case $m = 2$

Consider a path function $X(t) : \mathbb{R}_+ \to \mathbb{Z}_+$ s.t.:

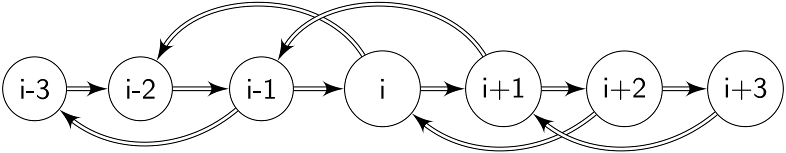

Figure: Path function $X(t)$.

Markov stationary process

Given $X(t)$ set its transition probability function

$$P_{i,j}(t) = Pr\left\{X(t+s) = j | X(s) = i\right\},$$

satisfying:

$$\left\{\begin{array}{rcl} P_{i,i}(\Delta t) & = & 1 - \lambda \Delta t + o(\Delta t), \\ P_{i,i+1}(\Delta t) & = & \lambda \Delta t + o(\Delta t), \end{array}\right. \quad \text{as } \Delta t \to 0, \text{ if } i < m.$$

$$\left\{\begin{array}{rcl} P_{i,i-m}(\Delta t) & = & \mu \Delta t + o(\Delta t) \\ P_{i,i}(\Delta t) & = & 1 - (\lambda + \mu) \Delta t + o(\Delta t), \quad \text{as } \Delta t \to 0, \text{ if } i \geq m. \\ P_{i,i+1}(\Delta t) & = & \lambda \Delta t + o(\Delta t), \end{array}\right.$$

Transition probability function properties

Set $\mathbf{P}(t)$ a infinite matrix function $\mathbf{P}(t) = [P_{i,j}(t)]_{(i,j)\in\mathbb{Z}_+^2}$

1. $\mathbf{P}(t) \geq 0$ and $\mathbf{P}(0) = \mathbb{I}$ (identity),
2. $\sum_{j=0}^{\infty} P_{i,j}(t) \equiv 1,\ t \in \mathbb{R}_+,\ n \in \mathbb{Z}_+$ (honesty),
3. $\mathbf{P}(s+t) = \mathbf{P}(s)\mathbf{P}(t)$, $(s,t) \in \mathbb{R}_+^2$ (Chapman–Kolmogorov),
4. $\lim_{t\to 0} \mathbf{P}(t) = \mathbb{I}$.

System of infinitely many differential equations

$$\begin{cases} \mathbf{P}'(t) = \mathbf{A}\,\mathbf{P}(t), \quad t \geq 0, \\ \\ \mathbf{P}(0) = \mathbb{I}, \end{cases}$$

$$\mathbf{A} = \begin{pmatrix} -\lambda & \lambda & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & -\lambda & \lambda & & 0 & 0 & 0 & 0 & \\ \vdots & \ddots & \ddots & \ddots & \vdots & & \vdots & \vdots & \\ \\ 0 & 0 & \cdots & 0 & -\lambda & \lambda & 0 & 0 & \cdots \\ \mu & 0 & \cdots & 0 & 0 & -(\lambda+\mu) & \lambda & 0 & \ddots \\ 0 & \mu & & 0 & 0 & 0 & -(\lambda+\mu) & \lambda & \ddots \\ \vdots & \ddots & \ddots & & \ddots & & \ddots & \ddots & \ddots \end{pmatrix}$$

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$$\begin{cases} \mathbf{P}'(t) = \mathbf{A}\,\mathbf{P}(t), \quad t \geq 0, \\ \mathbf{P}(0) = \mathbb{I}. \end{cases}$$

Recurrence relations

Set $\{Q_n\}_{n\in\mathbb{Z}_+}$ a sequence of polynomials generated by:

$$Q_n(x) = \frac{1}{\lambda^n}(\lambda + x)^n, \quad n = 0, \ldots, m,$$

$$(\lambda + \mu + x)Q_n(x) = \lambda Q_{n+1}(x) + \mu Q_{n-m}(x), \quad n \geq m.$$

Set $\{\mathbf{q}_r\}_{r\in\mathbb{Z}_+}$ with vector polynomials $\mathbf{q}_r = (q_{0,r}, \ldots, q_{m-1,r})$:

$$\mathbf{q}_{-1} = (0, 0, 0 \ldots, 0),\ \mathbf{q}_0 = (1, 0, \ldots, 0), \ldots, \mathbf{q}_{m-1} = (0, \ldots, 0, 1),$$

$$(\lambda + x)\mathbf{q}_r(x) = \mu\mathbf{q}_{r+m}(x) + \lambda\mathbf{q}_{r-1}(x), \quad r \in \{0, \ldots, m-1\}.$$

$$(\lambda + \mu + x)\mathbf{q}_r(x) = \mu\mathbf{q}_{r+m}(x) + \lambda\mathbf{q}_{r-1}(x), \quad r \in \{m, m+1, \ldots, \}.$$

Operator

$\mathbf{A} : \ell^2 \to \ell^2$

$$\mathbf{A}\mathbf{e}_j = \lambda \mathbf{e}_{j+1} - \lambda \mathbf{e}_j \quad j \in \{0, 1, 2, \dots, m-1\}$$

and

$$\mathbf{A}\mathbf{e}_j = \lambda \mathbf{e}_{j+1} - (\lambda + \mu)\mathbf{e}_j + \mu \mathbf{e}_{j-m}, \quad j \geq m,$$

with $\{\mathbf{e}_j\}_{j \in \mathbb{Z}_+}$ the standard basis of ℓ^2

$$\mathbf{A} = \begin{pmatrix} -\lambda & \lambda & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & -\lambda & \lambda & & 0 & 0 & 0 & 0 & \\ \vdots & \ddots & \ddots & \ddots & \vdots & & \vdots & \vdots & \\ \\ 0 & 0 & \cdots & 0 & -\lambda & \lambda & 0 & 0 & \cdots \\ \mu & 0 & \cdots & 0 & 0 & -(\lambda+\mu) & \lambda & 0 & \ddots \\ 0 & \mu & & 0 & 0 & 0 & -(\lambda+\mu) & \lambda & \ddots \\ \vdots & \ddots & \ddots & & \ddots & & \ddots & \ddots & \ddots \end{pmatrix}$$

Biorthogonality

Given $p(x) = b_\kappa \, x^\kappa + \cdots + b_1 \, x + b_0$, $p(\mathbf{A})$ denotes

$$p(\mathbf{A})\mathbf{u} = b_\kappa \mathbf{A}^\kappa \mathbf{u} + \cdots + b_1 \, \mathbf{A}\mathbf{u} + b_0 \mathbf{u}, \quad \mathbf{u} \in \ell^2.$$

Lemma

$$\left(\sum_{j=0}^{m-1} q_{j,r}(\mathbf{A}^\top)\mathbf{e}_j \right) \cdot Q_n(\mathbf{A})\mathbf{e}_0 = \delta_{n,r}, \quad (n,r) \in \mathbb{Z}_+^2.$$

Resolvent functions $j = 1, \ldots, m$

$$f_j(z, \mathbf{A}) = \left(z\mathbb{I} - \mathbf{A}^\top \right)^{-1} \mathbf{e}_{j-1} \cdot \mathbf{e}_0 \sim \sum_{\nu=0}^{\infty} \frac{(\mathbf{A}^\top)^\nu \mathbf{e}_{j-1} \cdot \mathbf{e}_0}{z^{\nu+1}}$$

Change of variables

$z = x + \lambda + \mu$

$$L_n(z) = \lambda^n Q_n(z - \lambda - \mu), \quad n \in \mathbb{Z}_+.$$

Then $\{L_n\}_{n\in\mathbb{Z}_+}$

$$L_n(z) = (z - \mu)^n, \quad n = 0, \ldots, m-1,$$

$$z\, L_n(z) = L_{n+1}(z) + \mu\lambda^m L_{n-m}, \quad n \geq m.$$

Consider $\{T_n\}_{n\in\mathbb{Z}_+}$ with

$$T_n(z) = z^n, \quad n = 0, \ldots, m-1.$$

$$z\, T_n(z) = T_{n+1}(z) + \mu\lambda^m T_{n-m}, \quad n \geq m,$$

Star system:

$$T_n(z) = z^n, \;\; n \in \{0, 1, \ldots, m\},$$

$$T_{n+1}(z) = zT_n(z) - \mu\lambda^m T_{n-m}(z), \quad n \geq m.$$







$\mathbf{T}$'s resolvent functions

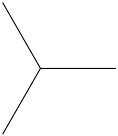
Theorem (Aptekarev, Kalyagin, Saff, Van Iseghem):

$$T_n(z) = z^n, \;\; n \in \{0, 1, \ldots, m\},$$

$$T_{n+1}(z) = zT_n(z) - \mu\lambda^m T_{n-m}(z), \quad n \geq m.$$

$$f_j(z, \mathbf{T}) = \frac{1}{\omega_0^j(z)} = \int_{S_0} \widetilde{\rho}_{j-1}(x)\, \widetilde{\rho}(x)\, \frac{d\,x}{z - x}, \quad j = 1, \ldots, m,$$

$$\omega_0^{m+1} - z\omega_0^m + \mu\lambda^m = 0 \quad \omega_0(z) = z + \mathcal{O}(1) \quad \text{as} \quad z \to \infty.$$



$$S_0 = \bigcup_{j=0}^{2} [0, a] e^{\frac{2\pi i}{3} j}, \qquad m = 2, \qquad a = \frac{(m+1)\sqrt[m+1]{\mu\lambda^m}}{m^{m/(m+1)}}.$$

Matrices

$m = 2$

$$f(\cdot, \mathbf{T}) \to f(\cdot, \mathbf{L}) \to f(\cdot, \mathbf{A})$$

$$\mathbf{L}^\top = \begin{pmatrix} \mu & 0 & \mu\lambda^2 & 0 & \cdots \\ 1 & \mu & 0 & \mu\lambda^2 & \\ 0 & 1 & 0 & 0 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix}, \ \mathbf{T}^\top = \begin{pmatrix} 0 & 0 & \mu\lambda^2 & 0 & \cdots \\ 1 & 0 & 0 & \mu\lambda^2 & \\ 0 & 1 & 0 & 0 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix}.$$

$$\mathbf{L} = \mathbf{T} + \mu\mathbb{I}_2, \quad \mathbf{A} = \mathbf{\Lambda}^{-1}\left[\mathbf{L} - (\lambda+\mu)\mathbb{I}\right]\mathbf{\Lambda},$$

$$\mathbb{I}_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \\ 0 & 0 & 0 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}, \quad \mathbf{\Lambda} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & \lambda & 0 & \\ 0 & 0 & \lambda^2 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}.$$

$\mathbf{L}$'s resolvent functions

In terms of $\mathbf{T}$'s

Lemma $(f(\cdot,\mathbf{T}) \to f(\cdot,\mathbf{L}))$

$$f_j(z,\mathbf{L}) = f_j(z,\mathbf{T}) + \left(\frac{1}{z-\mu}\right)^{j-1} - \frac{1}{z^{j-1}}.$$

Proof: $(\mathbf{e}_{j-1} \cdot \mathbf{T}^s \mathbf{e}_0 = \delta_{j-1,s})$.

$$f_j(z,\mathbf{L}) = \mathbf{e}_{j-1} \cdot [z\mathbb{I} - \mathbf{L}]^{-1}\,\mathbf{e}_0 = \mathbf{e}_j \cdot [z\mathbb{I} - \mathbf{T} + \mu\mathbb{I}_m]^{-1}\,\mathbf{e}_0$$

$$= \mathbf{e}_{j-1} \cdot [z\mathbb{I} - \mathbf{T}]^{-1} \left[\mathbb{I} + \mu\mathbb{I}_m\,(z\mathbb{I} - \mathbf{T})^{-1}\right]^{-1} \mathbf{e}_0$$

$$= f_j(z,\mathbf{T}) + \mathbf{e}_{j-1} \cdot \sum_{k=1}^{\infty} \sum_{s=0}^{m-1} \sum_{\tau=0}^{s} \frac{\mu^k}{z^k} \mathbb{I}_m \binom{s-\tau+k-1}{k-1} \frac{\mathbf{T}^s}{z^s} \mathbf{e}_0$$

$$= f_j(z,\mathbf{T}) + \frac{1}{z^{j-1}} \sum_{k=1}^{\infty} \binom{j-1+k}{k} \frac{\mu^k}{z^k} = f_j(z,\mathbf{T}) + \left(\frac{1}{z-\mu}\right)^{j-1} - \frac{1}{z^{j-1}}.$$

Matrices

$m = 2$

$$f(\cdot, \mathbf{T}) \to f(\cdot, \mathbf{L}) \to f(\cdot, \mathbf{A})$$

$$\mathbf{L}^\top = \begin{pmatrix} \mu & 0 & \mu\lambda^2 & 0 & \cdots \\ 1 & \mu & 0 & \mu\lambda^2 & \\ 0 & 1 & 0 & 0 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix}, \quad \mathbf{T}^\top = \begin{pmatrix} 0 & 0 & \mu\lambda^2 & 0 & \cdots \\ 1 & 0 & 0 & \mu\lambda^2 & \\ 0 & 1 & 0 & 0 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix}.$$

$$\mathbf{L} = \mathbf{T} + \mu\mathbb{I}_2, \quad \mathbf{A} = \mathbf{\Lambda}^{-1}\left[\mathbf{L} - (\lambda + \mu)\mathbb{I}\right]\mathbf{\Lambda},$$

$$\mathbb{I}_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \\ 0 & 0 & 0 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}, \quad \mathbf{\Lambda} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & \lambda & 0 & \\ 0 & 0 & \lambda^2 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}.$$

$\mathbf{A}$'s resolvent functions

Change of variable $z = x + \lambda + \mu$

Lemma $(f(\cdot, \mathbf{L}) \to f(\cdot, \mathbf{A}))$

$$f_j(x, \mathbf{A}) = \frac{1}{\lambda^{j-1}} f_j(x + \lambda + \mu, \mathbf{L}), \quad j \in \{1, \ldots, m\}$$

Proof.

$$f_j(x, \mathbf{A}) = \left[z\mathbb{I} - \mathbf{A}^\top\right]^{-1} \mathbf{e}_{j-1} \cdot \mathbf{e}_0$$

$$= \left[x\mathbb{I} - \mathbf{\Lambda}^{-1}\left(\mathbf{L}^\top - (\lambda + \mu)\mathbb{I}\right)\mathbf{\Lambda}\right]^{-1} \mathbf{e}_{j-1} \cdot \mathbf{e}_0$$

$$= \left[(x + \lambda + \mu)\,\mathbb{I} - \mathbf{L}^\top\right]^{-1} \mathbf{\Lambda}^{-1}\mathbf{e}_{j-1} \cdot \mathbf{\Lambda}\mathbf{e}_0$$

$$= \frac{1}{\lambda^{j-1}} \left[(x + \lambda + \mu)\,\mathbb{I} - \mathbf{L}^\top\right]^{-1} \mathbf{e}_{j-1} \cdot \mathbf{e}_0.$$

□

Proposition:

$$f_j(z,\mathbf{A}) = \int \frac{\left(\rho_{j-1}(u)\,\rho(u) + \frac{(-1)^j}{\lambda^{j-1}(j-2)!}(\delta_{-\lambda} - \delta_{-\mu-\lambda})^{(j-2)}(x)\right)du}{z-u}.$$

Proof.

$$f_j(z,\mathbf{A}) = \frac{1}{\lambda^{j-1}} \int \frac{\left(\widetilde{\rho}_{j-1}(x)\,\widetilde{\rho}(x) + \frac{(-1)^j}{(j-2)!}(\delta_{\mu} - \delta_0)^{(j-2)}(x)\right)dx}{z+\mu+\lambda-x}$$

$$\rho(u) = \widetilde{\rho}(u+\lambda+\mu) \quad \text{and} \quad \rho_{j-1}(u) = \frac{1}{\lambda^{j-1}}\widetilde{\rho}_{j-1}(u+\lambda+\mu).$$

□

Denote: $$\frac{d\sigma_j}{dx} = \rho_j(x)\rho(x) + \frac{(-1)^j}{\lambda^{j-1}(j-2)!}\left(\delta_{-\lambda} - \delta_{-\lambda-\mu}\right)^{(j-2)},$$

$$\text{with} \quad \rho_0 \equiv 1 \quad \text{and} \quad \frac{\delta^{(-1)}}{(-1)!} = 0.$$

Theorem

Biorthogonality (integral expression)

$$\delta_{n,r} = \int_{\Sigma_0 \cup \{-\lambda\}} Q_n(x) \sum_{j=0}^{m-1} q_{j,r}(x)\, d\sigma_j(x), \quad (n,r) \in \mathbb{Z}_+^2,$$

$$\Sigma_0 = \bigcup_{k=0}^{m} \left[-\lambda - \mu, -\lambda - \mu + \frac{m+1}{m} \left(\frac{\lambda \mu^m}{m} \right)^{1/(m+1)} \exp \frac{2\pi i k}{m+1} \right]:$$

Proof: Combining.

$$f_j(z, \mathbf{A}) \sim \sum_{\nu=0}^{\infty} \frac{(\mathbf{A}^\top)^\nu \mathbf{e}_{j-1} \cdot \mathbf{e}_0}{z^{\nu+1}} = \sum_{\nu=0}^{\infty} \frac{\int x^\nu \, d\sigma_j(x)}{z^{\nu+1}} \quad \text{and}$$

$$\left(\sum_{j=0}^{m-1} q_{j,r}(\mathbf{A}^\top) \mathbf{e}_j \right) \cdot Q_n(\mathbf{A}) \mathbf{e}_0 = \delta_{n,r}, \quad (n,r) \in \mathbb{Z}_+^2.$$

Theorem:

Main result

$$\mathbf{P}(t) = [P_{n,r}]_{(n,r)\in\mathbb{Z}_+^2} \text{ with } P_{n,r}(t) = \int e^{xt} Q_n(x) \sum_{j=0}^{m-1} q_{j,r}(x)\, d\sigma_j(x)$$

is the unique solution of $\mathbf{P}'(t) = \mathbf{A}\,\mathbf{P}(t)$, $t \geq 0$, $\mathbf{P}(0) = \mathbb{I}$, whose entries are transition probability functions:

1. $\mathbf{P}(t) \geq 0$ and $\mathbf{P}(0) = \mathbb{I}$ (b/c orthogonality and $(\lambda, \mu) \in \mathbb{R}_+^2$),
2. $\sum_{j=0}^{\infty} P_{i,j}(t) \equiv 1$, $t \in \mathbb{R}_+$, $n \in \mathbb{Z}_+$ (b/c $P_{i,j} \in \mathcal{H}(\mathbb{C})$),
3. $\mathbf{P}(s+t) = \mathbf{P}(s)\mathbf{P}(t)$, $(s,t) \in \mathbb{R}_+^2$ (b/c recurrence relations),
4. $\lim_{t\to 0} \mathbf{P}(t) = \mathbb{I}$ (b/c entries' continuity).

Expressions

$\omega_0, \ldots, \omega_m$ solutions of $\lambda\,\omega^{m+1} - (z+\lambda+\mu)\,\omega^m + \mu = 0$

$$Q_n(z) = \sum_{j=0}^{m} \frac{\left(\omega_j + \frac{\mu}{\lambda}\left(\frac{1}{\omega_j^m} - 1\right)\right)^m - 1}{\left(\omega_j^m - 1\right)\left(\omega_j^m - \frac{m\,\mu}{\lambda\,\omega_j}\right)}\,\omega_j^{m+n},$$

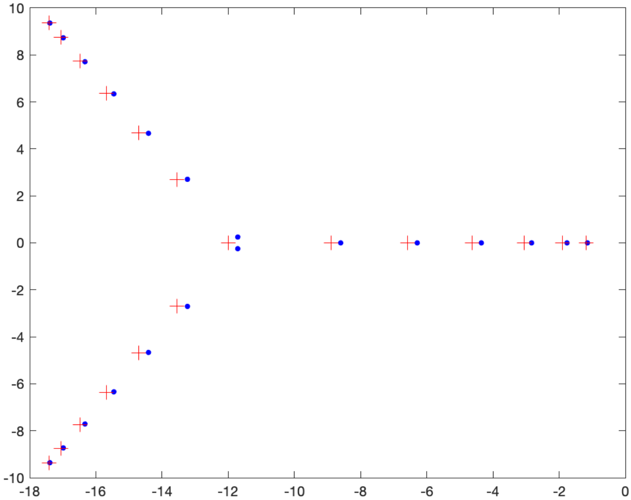
$$q_{j,r} = \frac{\lambda + x}{\mu}\delta_{j,r-m} - \frac{\lambda}{\mu}\delta_{j,r-m-1}, \quad j = 0, \ldots, m-1,$$

For $r \geq 2m$

$$q_{j,r} = \begin{cases} \displaystyle\sum_{k=0}^{m} \frac{\omega_k^m}{\omega^{r+1}} \frac{\omega_k^{m+1} + \frac{\mu}{\lambda}(1-\omega_k^m)}{\omega_k^{m+1} - \frac{m\mu}{\lambda}} & \text{if} \quad j = 0, \\[2ex] \displaystyle\sum_{k=0}^{m} \frac{\mu\omega_k^m}{\lambda\,\omega_k^{r+j+1}} \frac{1-\omega_k^m}{\omega_k^{m+1} - \frac{m\mu}{\lambda}} & \text{if} \quad j = 1, \ldots, m-1. \end{cases}$$

Q_{20}'s(blue dots) and $T_{20}(x+\mu+\lambda)$'s (red '+') roots

$m = 2$, $\lambda = 10$, and $\mu = 2$ (I thank Professor Calvetti)



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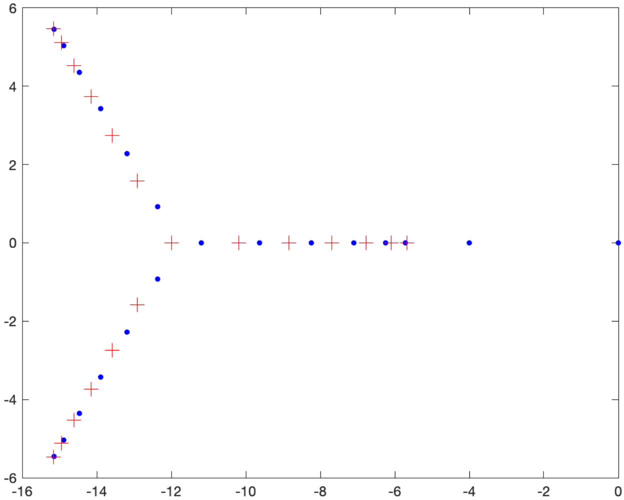

Congratulations Professor Littlejohns

Thank y'all so much