An approach to universality using Weyl *m*-functions

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joint work with
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Christoffel–Darboux kernel

- Let μ be a probability measure on $\mathbb{R}$ with all finite moments,

$$\int |\xi|^n \, d\mu(\xi) < \infty, \qquad \forall n \in \mathbb{N}.$$

  Assume that μ has infinite support (in sense of cardinality).
- We obtain orthonormal polynomials $\{p_j(z)\}_{j=0}^{\infty}$ by the Gram–Schmidt process from the sequence of monomials $\{z^j\}_{j=0}^{\infty}$ in $L^2(\mathbb{R}, d\mu)$.
- The Christoffel–Darboux (CD) kernel is

$$K_n(z, w) = \sum_{j=0}^{n-1} p_j(z)\overline{p_j(w)}.$$

  Reproducing kernel for subspace $\mathrm{span}\{1, z, \ldots, z^{n-1}\} \subset L^2(\mathbb{R}, d\mu)$.

Universality limits

- Universality limits of CD kernels are double scaling limits

$$\lim_{n\to\infty} \frac{1}{\tau_n} K_n \left(\xi + \frac{z}{\tau_n}, \xi + \frac{w}{\tau_n} \right)$$

for an appropriate sequence $\tau_n \to \infty$ and $z, w \in \mathbb{C}, \xi \in \mathbb{R}$.

- They are called universality limits because the limit is often found to be a standard kernel and does not depend on the exact measure we started with: the most common phenomenon is bulk universality, with limiting kernel

$$\frac{\sin \pi(\overline{w} - z)}{\pi(\overline{w} - z)}$$

- Interpretation: Kernel for Gaussian unitary ensemble, Paley Wiener Spaces, and ...

Local zero spacing

- Denote by $\xi_j^{(n)}$ for $j \in \mathbb{Z}$ the zeros of p_n counted from ξ, i.e.,

$$\cdots < \xi_{-2}^{(n)} < \xi_{-1}^{(n)} < \xi \leq \xi_0^{(n)} < \xi_1^{(n)} < \dots$$

- Freud–Levin theorem: The bulk universality limit implies

$$\lim_{n\to\infty} \tau_n(\xi_{j+1}^{(n)} - \xi_j^{(n)}) = 1 \qquad \forall j \in \mathbb{Z}.$$

  Statements of this type are commonly described as "clock behavior".
- Universality limits were first studied in the setting of random matrices, where this universal eigenvalue spacing was observed.

Previous results on bulk universality

$$\lim_{n\to\infty} \frac{K_n\left(\xi + \frac{z}{\tau_n}, \xi + \frac{w}{\tau_n}\right)}{\tau_n} = \frac{\sin(\pi(\overline{w} - z))}{\pi(\overline{w} - z)},$$

with approriate τ_n.

- 1971 Freud: on $[-1, 1]$ with $d\mu(\xi) = w(\xi)d\xi$ and strong conditions on w
- For Gaussian measure $d\mu(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\xi^2}\, d\xi$ follows from properties of Hermite polynomials
- 1990s Deift–Kriecherbauer–McLaughlin–Venakides–Zhou: Riemann–Hilbert techniques for measures

$$d\mu = e^{-Q(\xi)} d\xi$$

  Q a polynomial
- 2009 Lubinsky (Annals): Stahl–Totik regular measures $d\mu$ with local Lebesgue point/local Szegő conditions at ξ; with extensions by Findley, Simon and Totik.

A local criterion for bulk universality

Theorem (E.–Lukić–Simanek)

Let μ be a probability measure on $\mathbb{R}$ with infinite support and finite moments, corresponding to a determinate moment problem. Let

$$m(z) = \int \frac{1}{x-z}\, d\mu(x), \qquad z \in \mathbb{C}_+.$$

Let $\xi \in \mathbb{R}$ and assume that for some $0 < \alpha < \pi/2$,

$$f_\mu(\xi) := \frac{1}{\pi} \lim_{\substack{z\to\xi \\ \alpha \le \arg(z-\xi) \le \pi-\alpha}} \operatorname{Im} m(z) \in (0,\infty).$$

Then uniformly on compact regions of $(z,w) \in \mathbb{C}\times\mathbb{C}$,

$$\lim_{n\to\infty} \frac{K_n\left(\xi + \frac{z}{f_\mu(\xi)K_n(\xi,\xi)}, \xi + \frac{w}{f_\mu(\xi)K_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} = \frac{\sin(\pi(\overline{w}-z))}{\pi(\overline{w}-z)}.$$

Nontangential limits of $m(z)$

- The nontangential limit

$$f_\mu(\xi) := \frac{1}{\pi} \lim_{\substack{z \to \xi \\ \alpha \le \arg(z-\xi) \le \pi - \alpha}} \operatorname{Im} m(z)$$

exists for Lebesgue-a.e. $\xi \in \mathbb{R}$

- Pointwise, it exists at every Lebesgue point of the measure μ
- This limit recovers the a.c. part of the measure:

$$d\mu(\xi) = f_\mu(\xi) d\xi + d\mu_{\mathrm{s}}(\xi)$$

- The essential support for a.c. spectrum is the set

$$\Sigma_{\mathrm{ac}}(\mu) = \{\xi \in \mathbb{R} \mid f_\mu(\xi) \in (0, \infty)\}$$

In particular, this solves a conjecture of Avila–Last–Simon:

Corollary

Bulk universality holds almost everywhere on $\Sigma_{\mathrm{ac}}(\mu)$.

Rescaling and decoupling

- If $\lim_{n\to\infty} \frac{1}{\tau_n} K_n\left(\xi + \frac{z}{\tau_n}, \xi + \frac{w}{\tau_n}\right) = K(z,w)$, $K(0,0) \neq 0$, then

$$\lim_{n\to\infty} \frac{K_n(\xi,\xi)}{\tau_n} \in (0,\infty)$$

- Conversely, if

$$\lim_{n\to\infty} \frac{K_n(\xi,\xi)}{\tau_n} \in (0,\infty)$$

one scale can be replaced by the due to local uniform convergence.

- Christoffel functions for compactly supported measures:

$$\lim_{n\to\infty} \frac{K_n(\xi,\xi)}{n} = c > 0,$$

  1. μ is Stahl–Totik regular (global)
  2. local Szegő class condition (local)
  3. Lebesgue point conditions (local)

(Máté–Nevai–Totik 1991 Annals for $E = [-2,2]$, generalized by Totik)

Transfer matrices

- Define transfer matrices by

$$B(n,z) = A(a_n, b_n; z) \cdots A(a_1, b_1; z), \quad A(a_j, b_j; z) = \begin{pmatrix} \frac{z-b_j}{a_j} & -\frac{1}{a_j} \\ a_j & 0 \end{pmatrix}$$

- The Jacobi recursion

$$zp_n(z) = a_n p_{n-1}(z) + b_{n+1} p_n(z) + a_{n+1} p_{n+1}(z)$$

is equivalent to

$$\begin{pmatrix} p_n(z) \\ a_n p_{n-1}(z) \end{pmatrix} = B(n,z) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- Second kind polynomials are obtained by

$$\begin{pmatrix} q_n(z) \\ a_n q_{n-1}(z) \end{pmatrix} = B(n,z) \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Matrix CD kernel

- Matrix version of Christoffel–Darboux kernel defined by

$$\mathcal{K}_n(z,w) = \begin{pmatrix} \sum_{j=0}^{n-1} p_j(z)\overline{p_j(w)} & \sum_{j=0}^{n-1} q_j(z)\overline{p_j(w)} \\ \sum_{j=0}^{n-1} p_j(z)\overline{q_j(w)} & \sum_{j=0}^{n-1} q_j(z)\overline{q_j(w)} \end{pmatrix}$$

Note that $\mathcal{K}_n(z,w)_{11} = K_n(z,w)$

Limits of m-function

- We say $m : \mathbb{C}_+ \to \mathbb{C}_+$ has a normal limit at ξ, if

$$\eta = \lim_{y \downarrow 0} m(\xi + iy)$$

  Clearly $\eta \in \overline{\mathbb{C}_+} := \mathbb{C}_+ \cup \mathbb{R} \cup \{\infty\}$

- For $\eta \in \overline{\mathbb{C}_+}$, define

$$\mathring{H}_\eta := \frac{1}{1+|\eta|^2} \begin{pmatrix} 1 & -\operatorname{Re}\eta \\ -\operatorname{Re}\eta & |\eta|^2 \end{pmatrix} \quad \eta \in \mathbb{C}_+ \cup \mathbb{R}$$

$$\mathring{H}_\infty := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- Denote also $j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $h_\eta = \frac{\operatorname{Im}\eta}{1+|\eta|^2}$ and define

$$\mathring{\mathcal{K}}_\eta(z,w) = \frac{j(\cos(h_\eta(\overline{w}-z)-1)) + \frac{\mathring{H}_\eta}{h_\eta}\sin(h_\eta(\overline{w}-z))}{\overline{w}-z}$$

Bulk universality for matrix CD kernel

Theorem (E.–Lukić–Simanek)

Denote $\tau(n) = \operatorname{tr} \mathcal{K}_n(\xi,\xi)$*. The following are equivalent:*

1. *m has a normal limit at ξ,*

$$\lim_{y \downarrow 0} m(\xi + iy) = \eta \in \overline{\mathbb{C}_+}$$

2. *The (matrix) universality limit exists on the diagonal:*

$$\lim_{n\to\infty} \frac{1}{\tau(n)} \mathcal{K}_n(\xi,\xi) = H_\infty$$

3. *The (matrix) universality limit exists:*

$$\lim_{L\to\infty} \frac{1}{\tau(n)} \mathcal{K}_n\left(\xi + \frac{z}{\tau(n)}, \xi + \frac{w}{\tau(n)}\right) = \mathcal{K}_\infty(z,w).$$

Moreover, in this case, $H_\infty = \mathring{H}_\eta$ *and* $\mathcal{K}_\infty(z,w) = \mathring{\mathcal{K}}_\eta(z,w)$*.*

A connection to subordinacy theory

A special case of the previous theorem is a result from subordinacy theory: Using that

$$\tau(n) = \sum_{j=0}^{n-1} p_j(\xi)^2 + \sum_{j=0}^{n-1} q_j(\xi)^2$$

we get

$$\lim_{y \downarrow 0} m(\xi + iy) = \infty \qquad \Longleftrightarrow \qquad \lim_{n\to\infty} \frac{\sum_{j=0}^{n-1} p_j(\xi)^2}{\sum_{j=0}^{n-1} q_j(\xi)^2} = 0$$

Canonical systems

- Let $H : [0, \infty) \to \mathrm{Mat}(2, \mathbb{R})$ be locally integrable and
$$H(x) \geq 0,$$
for Lebesgue-a.e. x.
- Let $T : [0, \infty) \times \mathbb{C} \to \mathrm{Mat}(2, \mathbb{C})$ be the solution of the initial value problem
$$j\partial_x T(x, z) = -zH(x)T(x, z), \quad T(0, z) = I_2.$$
H is called the Hamiltonian of the canonical system.
- Assume the limit point case
$$\mathrm{tr} \int_0^\infty H(x)dx = \infty \tag{1}$$
- Due to (1) the Weyl m-function can be introduced. Let $\tau \in \overline{\mathbb{C}_+}$ and define
$$m(z) = \lim_{x \to \infty} T(x, z)^{-1} \star \tau$$
in the projective sense. This definition is independent of τ.

de Branges homeomorphism

- Define

$$\mathcal{K}_L(z,w) = \int_0^L T(x,w)^* H(x) T(x,z)\, dx,$$

and note that $\tau_L = \operatorname{tr} \mathcal{K}_L(0,0) = \operatorname{tr} \int_0^L H(x)\, dx$

The map

$$H \mapsto m$$

is onto the set $\{\text{Herglotz functions}\} \cup \{f(z) \equiv c : \quad c \in \mathbb{R} \cup \{\infty\}\}$ but not one-to-one. It is one-to-one up to "reparametrization" of $\mathbb{R}$.Thus:

- Canonical systems

$$j\partial_x T(x,z) = -zH(x)T(x,z), \qquad T(0,z) = I_2$$

- Reparametrize x to impose $\operatorname{tr} H = 1$ a.e.
- de Branges (uniqueness): map $H \mapsto m$ is a bijection
- The correspondences between $H, m, M, \mathcal{K}$ are homeomorphisms

Scaling operation

- Consider a trace-parametrized canonical system

$$j\partial_t T(t,z) = -zH(t)T(t,z), \qquad T(0,z) = I_2$$

  with Weyl function $m(z)$ and kernel $\mathcal{K}_t(z,w)$

- For $r > 0$, a scaling operation

$$\begin{aligned} m_r(z) &= m(z/r) \\ H_r(t) &= H(rt) \\ M_r(t,z) &= M(rt, z/r) \\ (\mathcal{K}_r)_t(z,w) &= \frac{1}{r}\mathcal{K}_{rt}(z/r, w/r) \end{aligned}$$

  found by Kasahara for Krein strings and used by Eckhardt–Kostenko–Teschl and Langer–Pruckner–Woracek for canonical systems to investigate large energy asymptotics of the m-function

- We use the scaling operation to “zoom in” towards $\xi \in \mathbb{R}$

Proofs of Theorems

Proof of Theorem 2: WLOG $\xi = 0$

- Start from transfer matrices $T(L, z)$ with Weyl function $m(z)$
- Consider family of canonical systems corresponding to Weyl functions

$$m_r(z) = \begin{cases} m(z/r) & r \in [1, \infty) \\ \eta & r = \infty \end{cases}$$

- Characterize continuity of this family in terms of $H, m, M, \mathcal{K}$

Proof of Theorem 1:

- In addition, use a translation trick and consider the family

$$\tilde{m}_r(z) = \begin{cases} m(z/r) - \operatorname{Re} m(i/r) & r \in [1, \infty) \\ if_\mu(0) & r = \infty \end{cases}$$

Thank you for your attention!