Zeros of Jacobi polynomials

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Joint work with Jorge Arvesú Carballo and Lance Littlejohn
started durng my semester at Baylor University in 2019

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$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\ P_m^{(\alpha,\beta)}(x)\ (1-x)^\alpha\ (1+x)^\beta dx = 0, \ \ m \neq n$$

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\ P_m^{(\alpha,\beta)}(x)\ (1-x)^\alpha\ (1+x)^\beta dx \neq 0, \ \ m = n$$

Zeros of Jacobi polynomials

Given any orthogonal sequence $\{p_n(x)\}_{n=0}^{\infty}$, the zeros (roots) of $p_n(x)$ and $p_{n-1}(x)$ are real, simple, interlacing, i.e., between each pair of consecutive zeros of $p_n(x)$, there is exactly one zero of $p_{n-1}(x)$ for each $n \in \mathbb{N}, n \geq 2$,

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For Jacobi polynomials, if $-1 < x_{1,n} < x_{2,n} < \cdots < x_{n,n} < 1$ are the n real, simple zeros of $P_n^{(\alpha,\beta)}(x)$, $\alpha, \beta > -1$, each of the $n-1$ open intervals $(x_{1,n}, x_{2,n}), \ldots, (x_{n-1,n}, x_{n,n})$ contains exactly one zero of $P_{n-1}^{(\alpha,\beta)}(x), n \geq 2$. Exactly the required number of open intervals for full interlacing.

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$$-1 < x_{1,n} < x_{1,n-1} < x_{2,n} < x_{2,n-1} < \cdots < x_{n-1,n-1} < x_{n,n} < 1$$

Interlacing of zeros of two polynomials of equal degree

If $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$ are the n real, simple zeros of $p_n(x)$

and

$y_{1,n} < y_{2,n} < \cdots < y_{n,n}$ are the n real, simple zeros of $q_n(x)$,

the zeros of $p_n(x)$ and $q_n(x)$ are interlacing if either

$$x_{1,n} < y_{1,n} < x_{2,n} < y_{2,n} < \cdots < x_{n,n} < y_{n,n}$$

or

$$y_{1,n} < x_{1,n} < y_{2,n} < x_{2,n} < \cdots < y_{n,n} < x_{n,n}$$

Jacobi polynomials, equal degree, different parameters

Askey 1990: Zeros of $P_n^{(\alpha,\beta)}(x), P_n^{(\alpha,\beta+1)}(x), \alpha > -1, \beta > -1$ interlace.

Askey Conjecture: Zeros $P_n^{(\alpha,\beta)}(x), P_n^{(\alpha,\beta+2)}(x), \alpha > -1, \beta > -1$ interlace.

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A more general result was proved by D-Jordaan-Mbuyi 2009

Zeros $P_n^{(\alpha,\beta)}(x)$, $P_n^{(\alpha-s,\beta+t)}(x)$ interlace $0 \leq s, t \leq 2, \alpha - s > -1, \beta > -1$.

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Proofs use "Mixed" Three Term Recurrence Relations that follow from contiguous relations for hypergeometric functions, a very powerful tool. **RED FLAG** We need $\alpha - s > -1$ for orthogonality since if $\alpha \approx -1$ and $s \approx 2$, $\alpha - s \approx -3$ and for $-3 <$ one parameter < -2, the Jacobi polynomial is quasi-orthogonal and not all its roots lie in $(-1, 1)$.

Open questions (Alan Sokal OPSFA 2019)

Suppose $\alpha > -1, \beta > -1$ and $n \in \mathbb{N}$.

Zeros of $P_n^{(\alpha,\beta)}(x)$ and $P_{n+1}^{(\alpha,\beta+1)}(x)$ interlacing?

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Interlacing for some $n \in \mathbb{N}$ but not every $n \in \mathbb{N}$? Breakdown (if it occurs) for small n or for large n? Breakdown "orderly" or "random"? Always partial and sometimes full interlacing?

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Theorem: If $\{z_i\}_{i=1}^{n+1}$ are the zeros of $P_{n+1}^{(\alpha,\beta+1)}(x)$, each of the $n+1$ open intervals $(-1,z_1),(z_1,z_2),\ldots,(z_n,z_{n+1})$ contains either the point l_n or exactly one (simple) zero of $P_n^{(\alpha,\beta)}(x)$ but not both.

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The zeros of $P_n^{(\alpha,\beta)}(x)$ and $P_{n+1}^{(\alpha,\beta+1)}(x)$ interlace

if and only if

$l_n < z_1$, where z_1 is the smallest zero of $P_{n+1}^{(\alpha,\beta+1)}(x)$.

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Always "partial" or "full" interlacing of roots of $P_n^{(\alpha,\beta)}(x), P_{n+1}^{(\alpha,\beta+1)}(x)$.

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Is it possible for $l_n \in (z_{n+1}, 1)$?

Zeros of $P_n^{(\alpha,\beta)}(x)$ and $P_{n+1}^{(\alpha,\beta+1)}(x), \alpha > -1, \beta > 0.$

Richard Wellman (Matlab): For $\alpha > -1, \beta > 0, n \in \mathbb{N}$,

$$l_n = -1 + \frac{2(n+1)(\alpha+n+1)}{(\alpha+\beta+2n+2)(\alpha+\beta+2n+3)} <$$

$$1 - \frac{2(\alpha+1)(\alpha+3)}{2n(n+\alpha+\beta+4)+(\alpha+3)(\alpha+\beta+3)}$$

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D-Jordaan proved that for $\alpha > -1, \beta > 0$, the largest zero z_{n+1} of $P_{n+1}^{(\alpha,\beta+1)}(x)$ satisfies $1 - \frac{2(\alpha+1)(\alpha+3)}{2n(n+\alpha+\beta+4)+(\alpha+3)(\alpha+\beta+3)} < z_{n+1}$

which means $z_{n+1} > l_n$ so $l_n \notin (z_{n+1}, 1)$.

Our result: Each interval $(-1, z_1), (z_1, z_2), \ldots, (z_n, z_{n+1})$ contains either l_n or one zero of $P_n^{(\alpha,\beta)}(x)$, where $\{z_i\}_{i=1}^{n+1}$ are the zeros of $P_{n+1}^{(\alpha,\beta+1)}(x)$.

Zeros of $P_n^{(\alpha,\beta)}(x)$ and $P_{n+1}^{(\alpha,\beta+1)}(x), \alpha > -1, \beta > 0.$

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Fix α, n. As $\beta \to \infty$, $l_n \to -1 + O(\frac{1}{\beta^2})$

Fix β, n. As $\alpha \to \infty$, $l_n \to -1 + O(\frac{1}{\alpha})$

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Fix α, β. As $n \to \infty$, $l_n \to -1 + \frac{2n^2}{4n^2} = -1/2.$

Zeros of $P_{n+1}^{(\alpha,\beta+1)}(x)$, $P_n^{(\alpha,\beta)}(x)$.

$$n = 7,\ \alpha = 6,\ \beta = 2.\ l_n = -0.63$$

$P_8^{(6,3)}(x)$	-0.87	-0.71	-0.51	-0.3	-0.0	0.2	0.5	0.7
$P_7^{(6,2)}(x)$	-0.89	-0.72	-0.49	-0.2	0.07	0.4	0.6	$--$

The zeros of $P_{n+1}^{(\alpha,\beta+1)}(x)$ and $P_n^{(\alpha,\beta)}(x)$ are not interlacing.

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The zeros of $P_{n+1}^{(\alpha,\beta+1)}(x)$ and $(x - l_n)P_n^{(\alpha,\beta)}(x)$ are interlacing.

Zeros of $P_{n+1}^{(\alpha+1,\beta+1)}(x)$ and $P_n^{(\alpha,\beta)}(x)$

For $n \geq 4$, at least $n-3$ of the $n-1$ intervals

with endpoints at successive zeros of $P_n^{(\alpha,\beta)}(x)$

contain a (simple) zero of $P_{n+1}^{(\alpha+1,\beta+1)}(x)$

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contain a (simple) zero of $P_{n+1}^{(\alpha+1,\beta+1)}(x)$

The location of the remaining 4 zeros of $P_{n+1}^{(\alpha+1,\beta+1)}(x)$??

A number of different possibilities can and do occur.

The calculations are extremely messy.

Jacobi equal degree, increase α, β each by 1.

Let $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ Jacobi polynomials with $\alpha > -1, \beta > -1$.

Assume $P_n^{(\alpha,\beta)}(x)$, $P_n^{(\alpha+1,\beta+1)}(x)$ have no common zeros.

If $\{x_i\}_{i=1}^{n}$ zeros of $P_n^{(\alpha,\beta)}(x)$, $-1 < x_1 < \cdots < x_n < 1$

at least $n-2$ zeros of $P_n^{(\alpha+1,\beta+1)}(x)$

lie between a pair of consecutive zeros of $P_n^{(\alpha,\beta)}(x)$.

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Remaining two zeros of $P_n^{(\alpha+1,\beta+1)}(x)$ either both lie in one of the intervals with endpoints at a pair of consecutive zeros of $P_n^{(\alpha,\beta)}(x)$ or one zero of $P_n^{(\alpha+1,\beta+1)}(x)$ lies in $(-1, x_1)$ and one zero of $P_n^{(\alpha+1,\beta+1)}(x)$ lies in $(x_n, 1)$.

Equal degree: Zeros of $P_n^{(\alpha,\beta)}(x)$, $P_n^{(\alpha+1,\beta+1)}(x)$.

Mixed TTRR has linear factor with root at $E = \frac{\alpha-\beta}{\alpha+\beta+2n+2}$

Let $x_1 < x_2 < \cdots < x_n$ be the zeros of $P_n^{(\alpha,\beta)}(x)$

Three possibilities:

(i) $-1 < E < x_1$. Zeros of $P_n^{(\alpha+1,\beta+1)}(x), P_n^{(\alpha,\beta)}(x)$ interlace

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(iii) $x_n < E < 1$. Zeros of $P_n^{(\alpha+1,\beta+1)}(x)$, $P_n^{(\alpha,\beta)}(x)$ interlace

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Never have full interlacing of zeros for every $n \in \mathbb{N}$ each $\alpha, \beta > -1$.

Full interlacing breaks down for n large.

Sharp results and Open Questions

D-Muldoon (2015) Let $\alpha > 0$, and $k, n \in \mathbb{N}, 1 \leq k \leq n-1$.

For each t, $0 \leq t \leq 2k$, excluding values of t for which $L_n^{\alpha}(x), L_{n-k}^{\alpha+t}(x)$ have a common root, the roots of $L_n^{\alpha}(x), L_{n-k}^{\alpha+t}(x)$ are interlacing.

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Interval $0 < t \leq 2k$ is largest possible for interlacing for **EVERY** $n \in \mathbb{N}$.

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Roots of $L_n^{\alpha}(x), L_{n-1}^{\alpha+t}(x)$ interlace for $0 \leq t \leq 2$ and $0 \leq t \leq 2$ is sharp.

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Pálmai (2013) "On the interlacing of cylinder functions":

Positive roots Bessel functions $J_\mu(x,)$ $J_{\mu+t}(x)$ interlace iff $0 < t \leq 2$.

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We "know" the sharp result for interlacing roots of equal degree Jacobi polynomials is $P_n^{(\alpha,\beta)}(x), P_n^{(\alpha-2,\beta+2)}(x), \alpha > 1, \beta > -1$. No proof yet.

Lance Littlejohn

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Thank you to the organizers of this conference, Fritz and Andrei.

Thank you for your attention