References

Based on the following papers and preprints:

- S. Denisov, On the existence of wave operators for some Dirac operators with square summable potentials, 2004.
- R. Bessonov, S. Denisov, A spectral Szegő theorem on the real line, 2020.
- R. Bessonov, S. Denisov, De Branges canonical systems with finite logarithmic integral, 2021.
- R. Bessonov, S. Denisov, Szegő condition, scattering, and vibration of Krein strings, preprint, 2022.

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Plan of the talk

1. The vibration of half-infinite inhomogeneous string: some numerics.
2. Krein's string.
3. Strings: Szegő condition on spectral measure and its characterizations, both through mass distribution and through the dynamics.
4. 1d Dirac.
5. How did we get there? Scattering for canonical systems: sharp results.

1. Non-homogeneous string and its vibration.

The free motion of the vibrating string on half-line with a given initial displacement u_0 is described by the solution $u = u(\xi, t)$ of the string equation

$$\begin{aligned} &\rho(\xi)u_{tt}(\xi,t) = u_{\xi\xi}(\xi,t), \\ &u(\xi,0) = u_0(\xi), \qquad\qquad \xi \in [0,\infty), \quad t \in \mathbb{R}_+, \\ &u_t(\xi,0) = u_\xi(0,t) = 0. \end{aligned}$$

This equation admits classical solution under the additional assumptions on the density ρ and u_0. For the homogeneous string with positive constant density ρ_0, the propagation of the wave with the initial profile u_0 has the well-known "traveling wave" form given by d'Alembert's formula:

$$u(\xi, t) = \frac{u_0(\xi + at) + u_0(\xi - at)}{2}, \qquad t \geqslant 0, \qquad a = \rho_0^{-1/2},$$

where we extended u_0 to the whole real line $\mathbb{R}$ as an even function. Moreover, if $u_0 \in L^2(\mathbb{R}_+)$, then

$$u(\xi, t) = F_{u_0}^{(0)}(\xi - at) + o(1), \qquad t \to +\infty, \tag{1}$$

for the function $F_{u_0}^{(0)} = 0.5 u_0(\xi) \in L^2(\mathbb{R})$, where the remainder "$o(1)$" is with respect to the $L^2(\mathbb{R}_+)$–norm.

What happens if the wave is non-homogeneous? Look at some numerics first.

Figure 1.

The first graph shows the density of the string. For each interval $[n, n+1] = E_n \cup F_n$, E_n carries the density 1, F_n carries the density 2, and $|F_n| \sim 1/\sqrt{n+1}$. As time increases, only a vanishing portion of the wave (shown in the red circle) propagates with the maximal speed.

Notice that

$$\sum_{n \geqslant 0} |F_n| = +\infty$$

which will make the difference later.

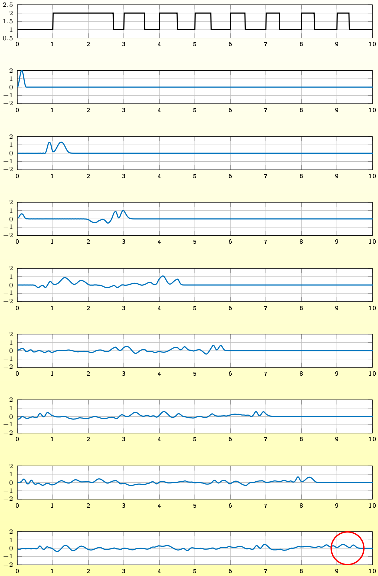

2.5 2 1.5 1 0.5
0 1 2 3 4 5 6 7 8 9 10
2 1 0 −1 −2
0 1 2 3 4 5 6 7 8 9 10

Figure 2.

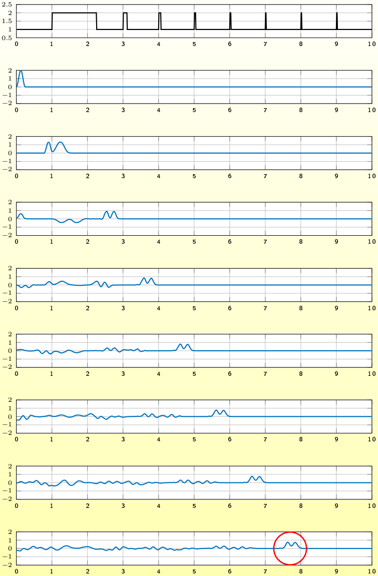
The first graph shows the density of the string. For each interval $[n, n+1] = E_n \cup F_n$, E_n carries the density 1, F_n carries the density 2. This time, $|F_n| \sim 1/(n+1)^2$. As time increases, a non-vanishing portion of the wave (shown in the red circle) propagates with the maximal speed.

Notice that

$$\sum_{n \geqslant 0} |F_n| < \infty$$

in this case and it seems the **"frontrunner does not fade away".**

Disclaimer: the times of snapshots in Figure 1 and Figure 2 are different.


2.5
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Can we formulate the sharp condition on ρ that will tell when the wave "propagates"? We will need some background to answer that question fully.

2. Krein strings.

To define the mathematical model of a vibrating string, one starts with prescribing its length $L \in (0, \infty]$ and the distribution function $M : [0, L) \to \mathbb{R}_+$. Given $\xi \in [0, L)$, the number $M(\xi)$ is interpreted as the **mass** of the $[0, \xi]$ piece. Define the Lebesgue–Stieltjes measure $\mathfrak{m}$ by $\mathfrak{m}[0, \xi] = M(\xi)$ and write its decomposition into the absolutely continuous and singular parts: $\mathfrak{m} = \mathfrak{m}_{\mathbf{ac}} + \mathfrak{m}_{\mathbf{s}} = \rho(\xi)d\xi + \mathfrak{m}_{\mathbf{s}}$. Denote $M(L-) = \lim_{\xi \to L, \xi < L} M(\xi)$. We will call the $[M, L]$ pair **proper** if M and L satisfy the following conditions

$$L + M(L-) = \infty,$$
$$0 < M(\xi) < M(L-), \quad \forall \xi \in (0, L).$$

The second condition can be interpreted as the string's left and right ends being "heavy".

The Krein's string is a non-negative self-adjoint differential operator, formally defined in the Hilbert space $L^2_{\mathfrak{m}}[0, L)$ by

$$\mathcal{S}_M = -\frac{d}{d\mathfrak{m}}\frac{d}{d\xi}, \qquad \xi \in [0, L), \tag{2}$$

with suitable boundary condition (we consider Neumann b.c. at zero). There is a canonical way to define the Weyl-Titchmarsh function q such that

$$q(z) = \int_{\mathbb{R}_+} \frac{d\sigma(x)}{x - z},$$

where σ is the **spectral measure** of the string $[M, L]$, a measure on $\mathbb{R}_+ = [0, +\infty)$ satisfying condition

$$0 < \int_{\mathbb{R}_+} \frac{d\sigma(x)}{1 + x} < \infty.$$

There is a bijection between strings and such σ. However, some classes of spectral measures are special.

3. Szegő class and its characterizations.

Definition. We say that $\sigma \in \mathrm{Sz}(\mathbb{R}_+)$ if $(x+1)^{-1} \in L^1(\sigma)$ and

$$\int_{\mathbb{R}_+} \frac{\log \sigma'(x)}{\sqrt{x}(x+1)}\, dx > -\infty.$$

We call it the **Szegő class** for Krein strings. It has important applications in Kolmogorov-Wiener theory of prediction for Gaussian stochastic processes. How can we characterize it?

(A). Direct characterization in terms of $[M, L]$, i.e., the string itself.

We need some notation first.

$$T^{(S)}(\xi) = \int_0^\xi \sqrt{\rho(s)}\, ds, \qquad L_\eta^{(S)} = \inf\{\xi : T^{(S)}(\xi) = \eta\},$$

for $\xi, \eta \in \mathbb{R}_+$. In physics literature, the former function is sometimes referred to as eikonal or optical metric.

Theorem (Direct characterization of Szegő measures for strings)

Let $[M, L]$ be a proper string, and let $\{\eta_n\}$ be an increasing sequence of positive numbers such that $c_1 \leqslant \eta_{n+1} - \eta_n \leqslant c_2$ for all $n \geqslant 0$ and some positive c_1 and c_2. Then, we have $\sigma \in \mathrm{Sz}(\mathbb{R}_+)$ if and only if $\sqrt{\rho} \notin L^1(\mathbb{R}_+)$ and

$$\sum_{n=0}^{+\infty} \Big(\Delta\xi_n \cdot \Delta M_n - (\Delta\eta_n)^2\Big) < +\infty\,, \qquad (3)$$

where $\xi_n = L_{\eta_n}^{(S)}$, $\Delta\xi_n = \xi_{n+2} - \xi_n$, $\Delta M_n = M(\xi_{n+2}) - M(\xi_n)$, $\Delta\eta_n = \eta_{n+2} - \eta_n$.

Some examples. In our first example, $\mathfrak{m}_{\mathbf{s}}(\mathbb{R}_+) = 0$ and the density ρ takes two positive values: a and b. So, we have

$$\rho(\tau) = \begin{cases} a, & \tau \in E, \\ b, & \tau \in \mathbb{R}_+ \setminus E, \end{cases} \tag{4}$$

for some σ-finite set $E \subseteq \mathbb{R}_+$ with respect to Lebesgue measure. We interpret such strings as those made of two types of material.

Lemma

The string with $\mathfrak{m}_{\mathbf{s}}(\mathbb{R}_+) = 0$ and density ρ of the form (4) has spectral measure σ in $\mathrm{Sz}(\mathbb{R}_+)$ iff either $a = b$ (the string is homogeneous) or one of the sets E, $\mathbb{R}_+ \backslash E$ has finite Lebesgue measure.

Hence, the string on Figure 1 is not in Szegő class, the string on Figure 2 is in Szegő class.

Lemma

Let $[M,\infty]$ be the string with $\mathfrak{m} = d\xi + \mathfrak{m}_{\mathbf{s}}$ on $\mathbb{R}_+$. Then, $\sigma \in \mathrm{Sz}(\mathbb{R}_+)$ iff $\mathfrak{m}_{\mathbf{s}}(\mathbb{R}_+) < \infty$.

(B). Dynamical characterization of the Szegő case, i.e., $\sigma \in \mathrm{Sz}(\mathbb{R}_+)$.

We look for the weak solution to the problem

$$\begin{aligned} &\mathfrak{m}(\xi)u_{tt}(\xi,t) = u_{\xi\xi}(\xi,t), \\ &u(\xi,0) = u_0(\xi), \qquad\qquad\qquad \xi \in [0,L), \quad t \in \mathbb{R}_+, \\ &u_t(\xi,0) = u_\xi(0,t) = 0. \end{aligned}$$

defined as $u = \cos(\sqrt{\mathcal{S}_M}t)u_0$ by the Spectral Theorem.

Recall that our topology is $L^2_{\mathfrak{m}}[0,L)$, i.e., $u, u_0 \in L^2_{\mathfrak{m}}[0,L)$.

For compactly supported initial data u_0, define the **front** of the propagating wave $u = u(\xi,t)$ at the moment t as

$$\mathfrak{fr}_t = \inf\left\{a \geqslant 0 : \, u(\xi,t) = 0, \forall \xi > a\right\} \, . \tag{5}$$

Theorem

Let $[M,L]$ be a proper string and let $u_0 \in L^2_{\mathfrak{m}}[0,L)$ be a nonzero compactly supported initial profile. Then, we have

$$\mathfrak{fr}_t = L^{(S)}_{\mathfrak{fr}_0+t},$$

for every $t>0$.

The theorem below provides a dynamical characterization of the Szegő class $\mathrm{Sz}(\mathbb{R}_+)$.

Theorem (Dynamical characterization of the Szegő class)

Let $[M, L]$ be a proper string and let σ be its spectral measure. Then $\sigma \in \mathrm{Sz}(\mathbb{R}_+)$ if and only if for some (and then for every) nonzero compactly supported initial profile $u_0 \in L^2(\mathfrak{m})$ and for some (and then for every) $\ell > 0$ we have

$$\limsup_{t\to+\infty} \|u(\cdot,t)\|_{L^2(\mathfrak{m},[\mathfrak{fr}_{t-\ell},\mathfrak{fr}_t])} > 0. \quad (6)$$

Put differently, the result says that the spectral measure of a string $[M, L]$ belongs to $\mathrm{Sz}(\mathbb{R}_+)$ if and only if the part of wave u near its wavefront does not vanish as $t \to +\infty$, i.e., "the frontrunner does not fade away".

To summarize, we obtained the following characterizations of Szegő class

$$\text{Condition (3)} \overset{\text{Direct}}{\Longleftrightarrow} \sigma \in \mathrm{Sz}(\mathbb{R}_+) \overset{\text{Dynamical}}{\Longleftrightarrow} \text{Condition (6)}$$

and that answers our original question.

Suppose $\sigma \in \mathrm{Sz}(\mathbb{R}_+)$. Can we say more about the asymptotics of u near its front? Yes, we skip formulation of the most general result and instead give an example. Recall the asymptotics for the free case, i.e., (see (1) above)

$$u(\xi,t) = F^{(0)}_{u_0}(\xi - at) + o(1), \qquad t \to +\infty, \quad a = \rho_0^{-1/2}.$$

Proposition

Let $[M,\infty]$ *be the string with* $\mathfrak{m} = d\xi + \mathfrak{m}_{\mathbf{s}}$ *on* $\mathbb{R}_+$, *and let* $u_0 \in L^2(\mathfrak{m})$ *have compact support,* $u_0 \neq 0$. *Then,*

$$\mathfrak{fr}_t = \mathfrak{fr}_0 + t, \qquad t \geqslant 0.$$

If $\mathfrak{m}_{\mathbf{s}}(\mathbb{R}_+) = \infty$, *we have* $\sigma \notin \mathrm{Sz}(\mathbb{R}_+)$ *and*

$$\lim_{t\to\infty} \|u\|_{L^2(\mathfrak{m},[\mathfrak{fr}_t - a, \mathfrak{fr}_t])} = 0,$$

for every $a > 0$. *In the case* $\mathfrak{m}_{\mathbf{s}}(\mathbb{R}_+) < \infty$, *we have* $\sigma \in \mathrm{Sz}(\mathbb{R}_+)$ *and*

$$\lim_{t\to+\infty} \|u(\xi,t)\|_{L^2(\mathfrak{m}_{\mathbf{s}},[\mathfrak{fr}_t - a, \mathfrak{fr}_t])} = 0,$$

$$\lim_{t\to+\infty} \|u(\xi,t) - F_{u_0}(\xi - t)\|_{L^2[\mathfrak{fr}_t - a, \mathfrak{fr}_t]} = 0,$$

for some $F_{u_0} \in L^2(\mathbb{R}), F_{u_0} \neq 0$, *and all* $a > 0$.

4. "Scattering for 1d Dirac" $\Leftrightarrow$ Szegő class. Consider Dirac operator

$$DX = J\partial X + \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix} X, \quad \tau \geqslant 0, \quad X_2(0) = 0, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

with spectral measure μ. Real-valued potential $q \in L^1_{\rm loc}(\mathbb{R}^+)$.

Theorem

If μ is the spectral measure for D, then the following conditions are equivalent

$$\sum_{n\geqslant 0} \left(\int_n^{n+2} h\, d\tau \cdot \int_n^{n+2} h^{-1} d\tau - 4 \right) < \infty,\ h(\tau) = e^{2\int_0^\tau q(s)\, ds}$$

$$\Updownarrow$$

$$\mu \in {\rm Sz}(\mathbb{R})_{\rm Dir},\ \textit{i.e.}\ (1+x^2)^{-1} \log \mu' \in L^1(\mathbb{R})$$

$$\Updownarrow$$

wave operators $W^{\pm}(D, D_0) = \lim_{t\to\pm\infty} e^{itD} e^{-itD_0}$ *exist*

5. Scattering for canonical systems. A Hamiltonian $\mathcal{H}$ on the positive half-axis $\mathbb{R}_+ = [0, +\infty)$ is a matrix-valued mapping of the form

$$\mathcal{H} = \begin{pmatrix} h_1 & h \\ h & h_2 \end{pmatrix}, \quad \operatorname{trace} \mathcal{H}(\tau) > 0, \quad \det \mathcal{H}(\tau) \geqslant 0, \quad \text{for a.e. } \tau \in \mathbb{R}_+.$$

The functions h_1, h_2, h are real-valued and belong to $L^1_{\mathrm{loc}}(\mathbb{R}_+)$. The Hilbert space $L^2(\mathcal{H})$ is given by the inner product

$$(X, Y)_{L^2(\mathcal{H})} = \int_0^\infty \langle \mathcal{H}(\tau) X(\tau), Y(\tau) \rangle_{\mathbb{C}^2} \, d\tau.$$

The Hamiltonian $\mathcal{H}$ gives rise to an eigenvalue problem

$$J \tfrac{\partial}{\partial \tau} \Theta(\tau, z) = z \mathcal{H}(\tau) \Theta(\tau, z), \; \Theta(0, z) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right), \; \tau \in \mathbb{R}_+, \; z \in \mathbb{C}$$

for the suitably defined self-adjoint differential operator $\mathcal{D}_{\mathcal{H}}$.

Important fact of Spectral Theory: in Hilbert space, every self-adjoint operator with simple spectrum can be modeled by $\mathcal{D}_{\mathcal{H}}$, i.e., it is "mother of all self-adjoint operators".

Our results include:

• Define the Szegő class for canonical systems by requiring that the associated spectral measure $\mu \in \mathrm{Sz}(\mathbb{R})$, i.e., it satisfies

$$\int_{\mathbb{R}} \frac{\log \mu'}{1+x^2} dx > -\infty \,.$$

We characterize all $\mathcal{H}$ for which this is true.

• The dynamical characterization of Szegő class is obtained.

• We study the long-time evolution of the group $e^{it\mathcal{D}_{\mathcal{H}}}$ and show that properly defined modified wave operators exist iff $\mu \in \mathrm{Sz}(\mathbb{R})$.

• The complete dynamical classification of all spectral types and the corresponding subspaces are obtained.

• Applications to Krein strings and 1d Dirac equations.