A new partition statistic

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Partitions

Definition

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Example

$p(5) = 7$: the partitions of (size) 5 are

$$(5)$$
$$(4, 1)$$
$$(3, 2)$$
$$(3, 1, 1)$$
$$(2, 2, 1)$$
$$(2, 1, 1, 1)$$
$$(1, 1, 1, 1, 1)$$

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Theorem (Euler, 1700s)

The generating function of $p(n)$:

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

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is essentially a weight $-1/2$ *modular form in terms of the Dedekind eta function*

$$\eta(z) := q^{1/24} \prod_{n \geq 1} (1-q^n) \,.$$

Known Results

Theorem (The Hardy–Ramanujan Asymptotic, 1918)

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}} \quad \textit{as } n \to \infty.$$

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Theorem (The Ramanujan Congruences, 1919)

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

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Theorem (Rademacher's absolutely convergent exact formula, 1943)

$$p(n) = \frac{2\pi}{(2n-1)^{3/4}} \sum_{c=1}^{\infty} \frac{A_c(n)}{c} I_{3/2}\left(\frac{\pi\sqrt{24n-1}}{6c}\right).$$

New Result

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Let $\lambda = \langle 1^{m_1}, 2^{m_2}, \ldots, r^{m_r} \rangle$ *be any partition. Then*

$$|\lambda| = \ell(\lambda)r + \sum_{i=3}^{r}(r-i)\ell\big(\lambda(1, i-1)\big) - rm_1,$$

where $\lambda(m,n)$ *is the subpartition of* λ *with parts* $m, m+1, \ldots, n$.

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Proof.

Iterated summation by parts and some cleverness. □

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- "partition Euler phi-function"

$$\varphi_{\mathcal{P}}(\lambda) := N(\lambda) \prod_{\substack{\lambda_i \in \lambda \\ \text{without repetition}}} (1 - \lambda_i^{-1})$$

where $N(\lambda) := \lambda_1 \lambda_2 \cdots \lambda_k$ is the **norm** of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$

The Norm Statistic

$$N(\lambda) := \lambda_1 \lambda_2 \cdots \lambda_k$$

The norm statistic has been investigated by

- MacMahon (1920s)
- Schneider–Sills
- Kumar–Rana

The Norm Statistic

To try to get an idea of $N(\lambda)$ compared to $|\lambda|$:

Theorem 2 (D–Just–Schneider, 2021)

Let $\lambda = \langle 1^{m_1}, 2^{m_2}, \ldots, r^{m_r} \rangle$, *and define* $\epsilon_r := \begin{cases} 1, & \text{if } r \text{ is odd} \\ 0, & \text{if } r \text{ is even} \end{cases}$. *Then*

$$\log N(\lambda) = \ell(\lambda) \log r + \epsilon_r m_{\lfloor r/2 \rfloor} \log\left(\left\lfloor \frac{r}{2} \right\rfloor\right) + \sum_{i=2}^{\lfloor r/2 \rfloor} \Big[\ell\big(\lambda(i, r-i+1)\big) - \ell\big(\lambda(1, r-i)\big)\Big] \log(r-i+1).$$

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$$\langle 1^{m_1}, 2^{m_2}, \ldots, k^{m_k} \rangle \mapsto \sum_{i=1}^{k} m_i \cdot i$$

$$N(\cdot) : \mathcal{P} \to \mathbb{N}, \text{ many partitions give factorizations of each } n$$

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Question

How can we change these maps to get more information about each n?

The Supernorm of a Partition

Definition

The **supernorm** of a partition $\lambda = \langle 1^{m_1}, 2^{m_2}, \ldots, k^{m_k} \rangle$ is

$$\widehat{N}(\lambda) := p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k} = \prod_{i=1}^{k} p_i^{m_i}$$

where p_i is the ith prime ($p_1 = 2$, $p_2 = 3$, $p_3 = 5$, etc.).

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$$\widehat{N}(\cdot) : \mathcal{P} \to \mathbb{N}, \text{ bijective map!}$$

$$\langle 1^{m_1}, 2^{m_2}, \ldots, k^{m_k} \rangle \mapsto \prod_{i=1}^{k} p_i^{m_i}$$

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Theorem 3 (D–Just–Schneider, 2021)

For any partition λ *with no 1s,* $\widehat{N}(\lambda) \leq N(\lambda)^{\log_2(3)}$.

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Proof.

For $\lambda = \langle 2^{m_2}, 3^{m_3}, \ldots, k^{m_k} \rangle$, rewrite

$$\widehat{N}(\lambda) = \prod_i p_i^{m_i} = \prod_i i^{m_i \log_i(p_i)}.$$

□

The Supernorm Statistic

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Proof.

- $p_a p_b > p_{a+b}$ for all positive integers a, b
- $\lambda = \langle 1^n \rangle$ maximizes $\widehat{N}(\lambda)$
- $\lambda = \langle n^1 \rangle$ minimizes $\widehat{N}(\lambda)$

□

Application to Partition Bijections

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Partition conjugation yields:

> *The number of partitions of size n and length k is equal to the number of partitions of size n with largest part k.*

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Apply $\widehat{N}$ to both sides:

> *Among the positive integers whose prime factors' indices sum to fixed $n \geq 1$, the number of integers with k prime factors is equal to the number of integers with largest prime factor p_k.*

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The set of integers with k prime factors and the set of integers with largest prime factor p_k both have arithmetic density zero.

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The set of squarefree integers has density $6/\pi^2$, but the set of integers with odd-indexed prime factors has density zero.

Arithmetic Densities Do Not Match

Observation

The densities of bijective subsets of $\mathbb{N}$ do not always match in the image of $\widehat{N}$, since $\widehat{N}$ scatters partitions of size n over a very large interval:

$$\text{(Theorem 4)} \qquad p_n \leq \widehat{N}(\lambda) \leq 2^n$$

Other Density Results

The supernorm can also yield other density results.

Theorem 5 (D–Just–Schneider, 2021)

Let $S \subseteq \mathbb{N}$, and let $d(S)$ be the arithmetic density of S. Then

$$-\sum_{\substack{n \geq 2 \\ \operatorname{index}(p_{\min}(n)) \in S}} \frac{\mu(n)}{n} = d(S).$$

Other Density Results

Proof of Theorem 5.

- If $p \in A \subseteq \mathbb{P} \iff \text{index}(p) \in S \subseteq \mathbb{N}$, then the natural density

$$d^*(A) = d(S).$$

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- Take $A = \{p_a : a \in S\}$ in Kural–McDonald–Sah:

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Note that $\widehat{N} : (k) \mapsto p_k$ gives $\mathbb{N} \supseteq S \longleftrightarrow A \subseteq \mathbb{P}$.

Hook lengths

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Example

Young diagram of $\gamma = (4,4,2,1,1)$ with hook lengths:

8	5	3	2
7	4	2	1
4	1		
2			
1			

t-Cores

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Example

$\gamma = (4, 4, 2, 1, 1)$ is 6-core and t-core for all $t \geq 9$.

8	5	3	2
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1			

2-Core Self-Conjugate Partitions

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Theorem

A self-conjugate partition is 2-core if and only if it is of the form

$$(k, k-1, \ldots, 3, 2, 1),$$

and

$$\mathrm{sc}_2(n) = \begin{cases} 1, & \text{if } n = \frac{k(k+1)}{2} \text{ for some } k \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

3-Core Self-Conjugate Partitions

Theorem (Robbins, 2000)

A self-conjugate partition of n *is 3-core if and only if it is of the form*

$$\langle 1^2, 2^2, \ldots, (r-2)^2, (r-1)^2, r, r+2, \ldots, 3r-4, 3r-2 \rangle$$

if $n = r(3r-2)$ *for some* $r \geq 1$, *or*

$$\langle 1^2, 2^2, \ldots, (r-1)^2, r^2, r+2, \ldots, 3r-2, 3r \rangle$$

if $n = r(3r+2)$ *for some* $r \geq 1$, *and*

$$\mathrm{sc}_3(n) = \begin{cases} 1, & \textit{if } n = r(3r \pm 2) \textit{ for some } r \geq 1 \\ 0, & \textit{otherwise.} \end{cases}$$

7-Core Self-Conjugate Partitions

Definition

For a discriminant $-D < 0$, the **Hurwitz class number** $H(-D)$ is the number of equivalence classes in $\mathrm{SL}_2(\mathbb{Z})$ of integral, binary quadratic forms of discriminant $-D$, weighted by $1/2$ the order of their automorphism group.

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Theorem (Ono–Raji, 2021)

If $n \not\equiv 5 \pmod 7$ is a positive odd integer, then

$$\mathrm{sc}_7(n) = \begin{cases} \frac{1}{4}H(-28n-56), & \text{if } n \equiv 1 \pmod 4 \\ \frac{1}{2}H(-7n-14), & \text{if } n \equiv 3 \pmod 8 \\ 0, & \text{if } n \equiv 7 \pmod 8. \end{cases}$$

7-Core Self-Conjugate Partitions

Theorem (Bringmann–Kane–Males, 2021)

For all $n \in \mathbb{N}$,

$$\mathrm{sc}_7(n) = \frac{1}{4}\Bigg(H(-28n-56) - H\left(\frac{-4n-8}{7}\right) - 2H(-7n-14) + 2H\left(\frac{-n-2}{7}\right)\Bigg).$$

Hook Length Formula

Theorem 6 (D–Sharp, 2021)

Let γ be a self-conjugate partition, and let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be the corresponding partition into distinct odd parts. Let i, j be integers such that (i, j) is a box in the Young diagram of γ, and assume $i \geq j$.

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$$h_\gamma(i,j) = h_\gamma(j,i) = \begin{cases} \frac{1}{2}(\lambda_i + \lambda_j), & \text{if } i \leq k \\ \frac{1}{2}(\lambda_j + 1) + j - i - 1 \\ \quad + \#\{j \leq m \leq k : \lambda_m \geq 2i - 2m + 1\}, & \text{if } i > k. \end{cases}$$

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Remark

Theorem 6 gives necessary and sufficient conditions for a self-conjugate partition to be t-core.

Interpretation of Hurwitz Class Numbers

Corollary 7 (D–Sharp, 2021)

Fix a positive integer n. Then

$$\frac{1}{4}\left(H(-29n-56)-H\left(\frac{-4n-8}{7}\right)-2H(-7n-14)+2H\left(\frac{-n-2}{7}\right)\right)$$

is equal to the number of squarefree natural numbers with prime factorizations $\prod_{1\leq i\leq k} p_{\lambda_i}$ whose indices λ_i are all odd and sum to n, such that $\frac{1}{2}(\lambda_i+\lambda_j)\not\equiv 0 \pmod 7$ for all pairs of integers (i,j) with $1\leq j\leq i\leq k$, and

$$\frac{1}{2}(\lambda_j+1)+j-i-1+\#\{j\leq m\leq k : \lambda_m\geq 2i-2m+1\}\not\equiv 0 \pmod 7$$

for all pairs of integers (i,j) with $1\leq j\leq k<i\leq \frac{1}{2}(\lambda_1+1)$ and $\lambda_j\geq 2i-2j+1$.

Summary

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The supernorm $\widehat{N}$ is a new partition statistic with many applications:

- Bijections between subsets of $\mathbb{N}$
- Densities of subsets of $\mathbb{N}$
- Combinatorial formula for Hurwitz class numbers
- Conjectures based on the additive-multiplicative correspondence
- More applications?