Quantum States in Random Environments

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The Anderson Model

The Anderson model is given by the operator family

$$[H_\omega \psi](n) = \sum_{\|m-n\|_1=1} \psi(m) + V_\omega(n)\psi(n)$$

in $\ell^2(\mathbb{Z}^d)$, where the potential $V_\omega(n)$ is given by independent identically distributed random variables.

It is used to model quantum states in the random environment specified by V_ω. Taking an initial state $\psi_0 \in \ell^2(\mathbb{Z}^d)$ with $\|\psi\|_2 = 1$, the time evolution of the state is governed by the Schrödinger equation

$$i\partial_t \psi = H_\omega \psi, \quad \psi|_{t=0} = \psi_0$$

and hence, by the spectral theorem, we have

$$\psi(t) = e^{-itH_\omega}\psi_0$$

The interpretation of this quantity is as follows:

$$\mathrm{Prob}\,(\text{state is at site } n \text{ at time } t) = \left|\langle \delta_n, e^{-itH_\omega}\psi_0\rangle\right|^2$$

The Anderson Model

The phenomenon commonly referred to as Anderson localization is given by the non-spreading of quantum states. For example, the statement

$$\sup_{t\in\mathbb{R}} \sum_{n\in\mathbb{Z}^d} |n|^p \left|\langle \delta_n, e^{-itH_\omega}\delta_0\rangle\right|^2 < \infty, \quad \text{a.e. } \omega$$

shows that the tails of the states must by super-polynomially small, uniformly in time. To distinguish it from a related concept discussed later, let us refer to it as dynamical localization.

A statement of this form is known in some cases, conjectured to hold in other cases, and conjectured to fail in yet some other cases. Specifically, it is known to hold for $d = 1$, conjectured to hold for $d = 2$, known to hold for arbitrary d for strong (and suitable) randomness, and conjectured to fail for $d \geq 3$ and small randomness.

A standard approach to studying questions about quantum dynamics proceeds via an analysis of the time-independent problem, that is, a study of the spectrum and the spectral measures of the random Schrödinger operator H_ω.

The Anderson Model: Known Results

Let us denote the single-site probability measure by ν and its topological support by $\operatorname{supp}\nu$. Assume that $\operatorname{supp}\nu$ is compact and contains at least two elements.

Theorem

For almost every ω, we have $\sigma(H_\omega) = \Sigma := [-2d, 2d] + \operatorname{supp}\nu$.

Theorem

Suppose that $d = 1$. Then, for almost every ω, H_ω has pure point spectrum with exponentially decaying eigenfunctions.

A result of this nature is commonly referred to as spectral localization. In this final form, the result is due to Carmona-Klein-Martinelli, but there were many prior important results, including those of Kunz-Souillard, Kotani, and Simon.

Theorem

Suppose that $d \in \mathbb{N}$ is arbitrary and ν is suitable. Then, there exists a non-trivial neighborhood Σ_{loc} of $\partial\Sigma$ such that for almost every ω, H_ω has pure point spectrum with exponentially decaying eigenfunctions in Σ_{loc}.

This result has a long history, with even more critical contributions. A very incomplete list is Fröhlich-Spencer, Fröhlich-Martinelli-Scoppola-Spencer, Aizenman-Molchanov, Simon-Wolff, Stollmann, Ding-Smart, among many others.

Beyond the Anderson Model: Presence of Correlations Between Sites

The fact that the potential values are i.i.d. is of critical importance to the proofs of the results mentioned above. It is of interest to include more general models in the discussion, where the underlying atomic structure may have i.i.d. features, but the potentials do not.

Let us propose a specific model. Given a single-site measure ν as above, let

$$\Omega = (\operatorname{supp} \nu)^{\mathbb{Z}^d}, \quad \mu = \nu^{\mathbb{Z}^d}, \quad f : \Omega \to \mathbb{R}$$

and, with $[T^n \omega](m) = \omega_{m+n}$ for $\omega \in \Omega$ and $m, n \in \mathbb{Z}^d$, let

$$V_\omega(n) = f(T^n \omega)$$

Note that the Anderson model corresponds to the choice $f(\omega) = \omega_0$.

As soon as f no longer depends on a single site, the potentials lose their i.i.d. structure and the proofs of Anderson localization (spectral and dynamical) break down. Nevertheless, the consideration of more general sampling functions f is natural and we ask whether similar spectral and quantum dynamical statements still hold.

Spectral Pseudorandomness: General Goals

Let us say that a model is spectrally pseudorandom if it shares essential spectral features with the Anderson model. For definiteness let us discuss this notion in one space dimension:

$$[H_\omega \psi](n) = \psi(n+1) + \psi(n-1) + V_\omega(n)\psi(n)$$

in $\ell^2(\mathbb{Z})$, where

$$V_\omega(n) = f(T^n \omega)$$

for a homeomorphism $T : \Omega \to \Omega$ on a compact metric space Ω and a continuous sampling function $f : \Omega \to \mathbb{R}$. We choose a T-ergodic Borel probability measure μ on Ω. Then there is a μ-almost sure spectrum Σ, and the spectral type of H_ω is μ-almost surely the same.

Note that the Anderson model arises via $\Omega = I^{\mathbb{Z}}$, T the shift, $f(\omega) = \omega_0$, $\mu = \nu^{\mathbb{Z}}$.

Spectral pseudorandomness asks, for example, for

- Σ to have only finitely many gaps
- H_ω to have pure point spectrum with exponentially decaying eigenfunctions for μ-a.e. ω

Spectral Pseudorandomness: Examples, Results, Conjectures

Consider the following three families of potentials:

$$V_\omega^{(\mathrm{qp})} = 2\lambda \cos(2\pi(n\alpha + \omega))$$
$$V_\omega^{(\mathrm{ss})} = 2\lambda \cos(2\pi(n^2\alpha + \omega))$$
$$V_\omega^{(\mathrm{dm})} = 2\lambda \cos(2\pi(2^n\omega))$$

The first family is known to not be pseudorandom: Σ is nowhere dense whenever $\lambda \neq 0$ and $\alpha \notin \mathbb{Q}$, and there are no eigenfunctions whenever $|\lambda| \leq 1$.

On the other hand, the second family and the third family are conjectured to be pseudorandom.

Note that the second family can be generated (modulo a minor modification) by the skew shift $T^{(\mathrm{ss})} : \mathbb{T}^2 \to \mathbb{T}^2$, $(\omega_1, \omega_2) \mapsto (\omega_1 + \alpha, \omega_1 + \omega_2)$ and the third family can be generated by the doubling map $T^{(\mathrm{dm})} : \mathbb{T} \to \mathbb{T}$, $\omega \mapsto 2\omega$.

Partial results in support of these conjectures are due to Chulaevsky-Spencer, Bourgain-Schlag, D.-Killip, Krüger, Han-Lemm-Schlag, among others.

The Almost Sure Spectrum

Let us consider an ergodic family of Schrödinger operators $\{H_\omega\}_{\omega\in\Omega}$ in $\ell^2(\mathbb{Z})$ as above and, for some T-ergodic measure μ, the associated almost sure spectrum Σ.

The density of states measure $\kappa = \kappa_{\mu,f}$ associated with the family $\{H_\omega\}_{\omega\in\Omega}$ is given by

$$\int g \, d\kappa = \int_\Omega \langle \delta_0, g(H_\omega)\delta_0 \rangle \, d\mu(\omega)$$

In other words, κ is the μ-average of the spectral measure corresponding to the pair (H_ω, δ_0). The accumulation function of κ,

$$k(E) = k_{\mu,f}(E) = \int \chi_{(-\infty,E]} \, d\kappa$$

is called the integrated density of states (IDS).

It is not hard to see that κ is a probability measure whose topological support coincides with Σ. Thus k is an increasing function that grows precisely at points in Σ and takes the value 0 below Σ and the value 1 above Σ.

In particular, k is constant on each gap of Σ and each gap can be labeled uniquely by the value k takes on it.

Gap Labelling in One Dimension via the Schwartzman Group

Gap labelling theory attempts to characterize these gaps labels in useful ways. The classical approach to gap labelling, developed by Bellissard et al., is based on K-theory of C^*-algebras. The advantage of this approach lies in its scope, as it applies in arbitrary dimensions and to operators that are more general than Schrödinger operators. The disadvantage is that actual computations of gap labels are very difficult.

For Schrödinger operators in one dimension, there is an alternative approach due to Johnson. While its scope is much more restricted, actual computations of gap labels are far easier.

Given a topological dynamical system (Ω, T) as above, we define the suspension of the dynamics $(X, \overline{T})$ as follows: X is the quotient of $\Omega \times [0,1]$ modulo the equivalence relation $(\omega, 1) \sim (T\omega, 0)$ and $\overline{T}^t \cdot [\omega, s] = [\omega, s + t]$.

If μ is a T-ergodic Borel probability measure on Ω, we define the suspension of the measure $\overline{\mu}$ on X by

$$\int_X f \, d\overline{\mu} = \int_0^1 \int_\Omega f([\omega, s]) \, d\mu(\omega) \, ds$$

Gap Labelling in One Dimension via the Schwartzman Group

Let $C^\sharp(X,\mathbb{T})$ be the set of equivalence classes in $C(X,\mathbb{T})$ modulo homotopy; $C^\sharp(X,\mathbb{T})$ is a countable abelian group (with group operation $[\phi_1]+[\phi_2]=[\phi_1+\phi_2]$).

Given $\phi \in C(X,\mathbb{T})$ and $x \in X$, we obtain a continuous function $\phi_x : \mathbb{R} \to \mathbb{T}$ by following the image of ϕ along the orbit of x, $\phi_x(t) = \phi(\overline{T}^t x)$.

With the canonical projection $\pi : \mathbb{R} \to \mathbb{T}$, we observe that for each $a \in \pi^{-1}\{\phi_x(0)\}$, there is a unique continuous lift $\widetilde{\phi}_x : \mathbb{R} \to \mathbb{R}$ that satisfies $\pi \circ \widetilde{\phi}_x = \phi_x$ and $\widetilde{\phi}_x(0) = a$.

Proposition

(a) *For each $\phi \in C(X,\mathbb{T})$, the limit*

$$\mathrm{rot}(\phi; x) = \lim_{t\to\infty} \frac{\widetilde{\phi}_x(t)}{t}$$

exists for $\overline{\mu}$-almost every $x \in X$ and does not depend on the choice of lift. Moreover, $\mathrm{rot}(\phi;x)$ is $\overline{\mu}$-almost surely independent of x and hence may be denoted by $\mathfrak{A}_{\overline{\mu}}(\phi)$.

(b) *If ϕ and ϕ' are homotopic, then $\mathfrak{A}_{\overline{\mu}}(\phi) = \mathfrak{A}_{\overline{\mu}}(\phi')$.*

The induced map

$$\mathfrak{A}_{\overline{\mu}} : C^\sharp(X, \mathbb{T}) \to \mathbb{R}$$

is called the Schwartzman homomorphism.

Theorem (Johnson's Gap-Labelling Theorem)

Given an ergodic topological dynamical system (Ω, T, μ) *with* $\operatorname{supp} \mu = \Omega$*, let* $(X, \overline{T}, \overline{\mu})$ *denote its suspension, let* $C^\sharp(X, \mathbb{T})$ *denote the set of homotopy classes of maps* $X \to \mathbb{T}$*, let* $\mathfrak{A}_{\overline{\mu}} : C^\sharp(X, \mathbb{T}) \to \mathbb{R}$ *denote the Schwartzman homomorphism, and denote by*

$$\mathfrak{A} = \mathfrak{A}(\Omega, T, \mu) := \mathfrak{A}_{\overline{\mu}}(C^\sharp(X, \mathbb{T}))$$

the range of the Schwartzman homomorphism. Then, for any continuous $f \in C(\Omega, \mathbb{R})$*,*

$$k_{\mu,f}(E) \in \mathfrak{A} \cap [0, 1]$$

for all $E \in \mathbb{R} \setminus \Sigma_{\mu,f}$*.*

Gap Labelling: Absence of Interior Gaps

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Gap Labelling: Absence of Interior Gaps

Theorem (D.-Fillman)

Consider $T_{A,b} : \mathbb{T}^d \to \mathbb{T}^d$, $\omega \mapsto A\omega + b$, *where* $A \in \mathrm{SL}(d, \mathbb{Z})$, $b \in \mathbb{T}^d$. *Suppose* μ *is* $T_{A,b}$*-ergodic with* $\mathrm{supp}(\mu) = \mathbb{T}^d$. *Then,*

$$\mathfrak{A}(\mathbb{T}^d, T_{A,b}, \mu) = \{\langle k, b\rangle + n : n \in \mathbb{Z} \text{ and } k \in \mathbb{Z}^d \cap \ker(I - A^*)\}$$

Remark

(a) Note that $\mathfrak{A}(\mathbb{T}^d, T_{A,b}, \mu) = \mathbb{Z}$ if 1 is not an eigenvalue of A^*.

(b) If $\mathfrak{A}(\mathbb{T}^d, T_{A,b}, \mu) = \mathbb{Z}$, then $\Sigma_{f,\mu}$ is connected, it has no gaps!

(c) A prominent example is given by the cat map, for which $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $b = 0$. In this case, $\mathfrak{A} = \mathbb{Z}$.

(d) Another prominent example is the skew shift, for which $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$. In this case, $\mathfrak{A} = \mathbb{Z} + \alpha\mathbb{Z}$.

Gap Labelling: Absence of Interior Gaps

Recall that the doubling map model, with potential given by

$$V_\omega^{(\mathrm{dm})} = 2\lambda \cos(2\pi(2^n\omega))$$

is one of the prominent examples for which spectral pseudorandomness is expected.

Developing the result for affine torus maps above further, we can confirm one of the spectral pseudorandomness aspects in full generality:

Theorem (D.-Fillman)

Suppose $\Omega = \mathbb{T}$, $T\omega = 2\omega$, $\mu = \mathrm{Leb}$, and $f \in C(\mathbb{T}, \mathbb{R})$. Then, the almost surely common essential spectrum $\Sigma_{f,\mu}$ of the associated Schrödinger operators in $\ell^2(\mathbb{Z}_+)$ is connected.

Gap Labelling: Generic Gap Opening

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Gap Labelling: Generic Gap Opening

Recall that Johnson's gap-labelling theorem shows that for any continuous $f \in C(\Omega, \mathbb{R})$,

$$k_{\mu,f}(E) \in \mathfrak{A} \cap [0,1]$$

for all $E \in \mathbb{R} \setminus \Sigma_{\mu,f}$. Two natural questions:

1. Is this collection of gap labels minimal, that is, are they all needed?
2. If a computation of $\mathfrak{A}$ yields a dense set, can one use this to show Cantor spectrum, that is, that the gaps are dense for some $f \in C(\Omega, \mathbb{R})$?

Theorem (Avila-Bochi-D.)

Suppose Ω is a compact metric space and T is a strictly ergodic homeomorphism that has a non-periodic finite-dimensional factor. Then for each $\ell \in \mathfrak{A} \cap [0,1]$, the set

$$\{f \in C(\Omega, \mathbb{R}) : \Sigma_{\mu,f} \text{ has an open gap with label } \ell\}$$

is open and dense. In particular, for a generic $f \in \mathbb{C}(\Omega, \mathbb{R})$, all gaps allowed by the gap labelling theorem are open.

Note that this result is somewhat at odds with one of the conjectured spectral pseudorandomness features of the specific skew-shift potential

$$V_\omega^{(\mathrm{ss})} = 2\lambda \cos(2\pi(n^2\alpha + \omega))$$

Anderson Localization in the Presence of Correlations

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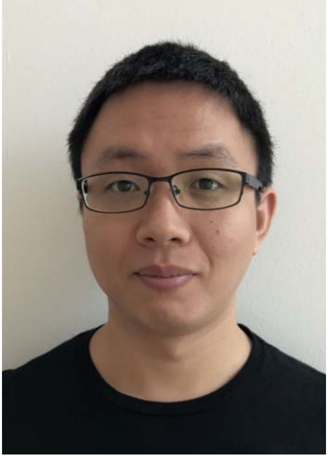

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Anderson Localization in the Presence of Correlations

Let us discuss the model arising from the Anderson model by allowing more general sampling functions proposed earlier; for $d = 1$. For simplicity, let us base it on the Bernoulli case:

$$\Omega = \{0,1\}^{\mathbb{Z}}, \quad [T\omega]_n = \omega + 1, \quad \mu = \nu^{\mathbb{Z}}, \quad f : \Omega \to \mathbb{R}$$

where $\nu(\{0\}) = 1 - \nu(\{1\}) = p \in (0,1)$. Our goal is to prove (spectral and/or dynamical) localization.

It was shown about 20 years ago by Bourgain, Goldstein, and Schlag that localization proofs can be based on two key ingredients:

- positivity of the Lyapunov exponent
- a large deviation estimate for the Lyapunov exponent

These statements concern the energy-dependent skew products

$$(T, A_E) : \Omega \times \mathbb{R}^2 \to \Omega \times \mathbb{R}^2, \quad (\omega, v) \mapsto (T\omega, A_E(\omega)v)$$

with

$$A_E : \Omega \to \mathrm{SL}(2,\mathbb{R}), \quad \omega \mapsto \begin{pmatrix} E - f(\omega) & -1 \\ 1 & 0 \end{pmatrix}$$

and the iterates $(T, A_E)^n = (T^n, A_E^n)$.

Anderson Localization in the Presence of Correlations

Kingman's subadditive ergodic theorem yields the existence of the (non-negative!) Lyapunov exponent

$$L(E) = \lim_{n\to\infty} \frac{1}{n} \log \|A_E^n(\omega)\| \quad \mu - \text{a.e. } \omega \in \Omega$$

Given an interval $I \subseteq \mathbb{R}$, we say that PLE holds on I if

$$\inf_{E \in I} L(E) > 0$$

We say that ULD holds on I if for every $\varepsilon > 0$ there are $C, c > 0$ such that for each $E \in I$ and $n \in \mathbb{N}$, we have

$$\mu\left(\left\{\omega \in \Omega : \left|\tfrac{1}{n} \log \|A_E^n(\omega) - L(E)\right| > \varepsilon\right\}\right) < Ce^{-cn}$$

In the case of the Anderson model, the proof of the first step relies on Furstenberg's theorem, which in turn requires that the potential values are i.i.d.

Finding a way to prove the first step in our setting is the most challenging aspect of this extension.

Anderson Localization in the Presence of Correlations

Denote

$$\mathcal{Z} := \{E : L(E) = 0\}$$

Theorem (Avila-D.-Zhang)

Suppose f is Hölder continuous and non-constant. Then $\mathcal{Z}$ is discrete.

Theorem (Avila-D.-Zhang)

Suppose f is locally constant and non-constant. Then $\mathcal{Z}$ is finite.

Let us sketch the proof of the second theorem.

Definition

The local stable set of a point $\omega \in \Omega$ is defined by

$$W^s_{\mathrm{loc}}(\omega) = \{\tilde{\omega} \in \Omega : \omega_n = \tilde{\omega}_n \text{ for } n \geq 0\}$$

and the local unstable set of ω is defined by

$$W^u_{\mathrm{loc}}(\omega) = \{\tilde{\omega} \in \Omega : \omega_n = \tilde{\omega}_n \text{ for } n \leq 0\}$$

Anderson Localization in the Presence of Correlations

By linearity and invertibility of each $A_E(\omega)$, we can projectivize the second component and consider

$$(T, A_E) : \Omega \times \mathbb{RP}^1 \to \Omega \times \mathbb{RP}^1$$

Let us denote the fiber $\{\omega\} \times \mathbb{RP}^1$ by $\mathcal{E}_\omega$.

Definition

A stable holonomy h^s for A is a family of homeomorphisms $h^s_{\omega,\omega'} : \mathcal{E}_\omega \to \mathcal{E}_{\omega'}$, defined whenever ω and ω' belong to the same local stable set, satisfying

(i) $h^s_{\omega',\omega''} \circ h^s_{\omega,\omega'} = h^s_{\omega,\omega''}$ and $h^s_{\omega,\omega} = \mathrm{id}$,

(ii) $A(\omega') \circ h^s_{\omega,\omega'} = h^s_{T\omega,T\omega'} \circ A(\omega)$,

(iii) $(\omega,\omega') \mapsto h^s_{\omega,\omega'}(\phi)$ is continuous when ω, ω' belong to the same local stable set, uniformly in ϕ.

An unstable holonomy $h^u_{\omega,\omega'} : \mathcal{E}_\omega \to \mathcal{E}_{\omega'}$ is defined analogously for pairs of points in the same unstable set.

If f is locally constant, then the limits

$$H^s_{\omega,\omega'} = \lim_{n\to\infty} A^n_E(\omega')^{-1} A^n_E(\omega), \quad H^u_{\omega,\omega'} = \lim_{n\to\infty} A_E^{-n}(\omega')^{-1} A_E^{-n}(\omega)$$

exist for ω, ω' in the same stable (resp., unstable) set. Projectivizing them yields stable and unstable holonomies.

Anderson Localization in the Presence of Correlations

Proposition (Bonatti-Gómez-Mont-Viana)

If $L(E) = 0$, then every (T, A_E)-invariant probability measure m on $\Omega \times \mathbb{RP}^1$ that projects to μ in the first component has a continuous disintegration $\{m_\omega : \omega \in \Omega\}$ that is invariant under the stable and unstable holonomies.

To conclude the proof, we proceed as follows:

- We pass to the conformal barycenter c_ω of each m_ω.
- We use cocycle invariance to identify c_ω with the Weyl-Titchmarsh m-function for each periodic point ω.
- We use holonomy invariance to relate c_ω and $c_{\omega'}$ for pairs (ω, ω') of periodic points. This yields an analytic relation that holds at all energies $E \in \mathcal{Z}$.
- If $\mathcal{Z}$ is infinite, then the relation extends to all E.
- If the relation holds for all E, the spectra of H_ω and $H_{\omega'}$ coincide.
- However, if f is also non-constant, then there exists a pair (ω, ω') of periodic points for which the spectra of H_ω and $H_{\omega'}$ do not coincide; contradiction.