Antisymmetry relations for continuous q-Jacobi polynomials

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Adopted notation

- $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, ...\}$; $(a)_n := a(a+1)\cdots(a+n-1)$
- $\pm a := \{a, -a\}$, $z^{\pm} := \{z, z^{-1}\}$; $(a;q)_n := (1-a)(1-qa)\cdots(1-q^{n-1}a)$
- Nonterminating generalized hypergeometric series (product notation)

$$ {}_rF_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; z\right) := \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r)_k}{(b_1, \ldots, b_s)_k} \frac{z^k}{k!}, $$

- Nonterminating and basic hypergeometric series

$$ {}_{r+1}\phi_s\left(\begin{matrix} a_1, ..., a_{r+1} \\ b_1, ..., b_s \end{matrix}; q, z\right) := \sum_{k=0}^{\infty} \frac{(a_1, ..., a_{r+1}; q)_k}{(q, b_1, ..., b_s; q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{s-r} z^k $$

- Nonterminating very-well-poised basic hypergeometric series

$$ {}_{r+1}W_r(a; a_4, \ldots, a_{r+1}; q, z) := {}_{r+1}\phi_r\left(\begin{matrix} a, \pm q\sqrt{a}, a_4, \ldots, a_{r+1} \\ \pm\sqrt{a}, \frac{qa}{a_4}, \ldots, \frac{qa}{a_{r+1}} \end{matrix}; q, z\right) $$

Jacobi and Gegenbauer polynomials and functions

Jacobi polynomial, is defined as

$$P_n^{(\alpha,\beta)}(x) := \frac{\Gamma(\alpha+1+n)}{n!\Gamma(\alpha+1)}\,{}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix};\frac{1-x}{2}\right).$$

The Gegenbauer (or ultraspherical) polynomial is defined as

$$C_n^\mu(x) = \frac{(2\mu)_n}{(\mu+\frac12)_n}P_n^{(\mu-\frac12,\mu-\frac12)}(x) = \frac{\Gamma(2\mu+n)}{n!\Gamma(2\mu)}\,{}_2F_1\left(\begin{matrix}-n, 2\mu+n\\ \mu+\frac12\end{matrix};\frac{1-x}{2}\right).$$

The Jacobi function of the first kind (Durand, Koornwinder, Koelink, ...)

$$P_\gamma^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)\Gamma(\alpha+1)}\,{}_2F_1\left(\begin{matrix}-\gamma, \alpha+\beta+\gamma+1\\ \alpha+1\end{matrix};\frac{1-x}{2}\right).$$

The Gegenbauer function of the first kind is defined as

$$C_\gamma^\mu(x) = \frac{\Gamma(2\mu+\gamma)}{\Gamma(\gamma+1)\Gamma(2\mu)}\,{}_2F_1\left(\begin{matrix}-\gamma, 2\mu+\gamma\\ \mu+\frac12\end{matrix};\frac{1-x}{2}\right).$$

Jacobi polynomials: $P_n^{(\alpha,\beta)} : \mathbb{C} \to \mathbb{C}$

- Jacobi polynomials are orthogonal polynomials, orthogonal on the real interval $(-1,1)$. They can be defined for $n \in \mathbb{N}_0$ as

$$P_n^{(\alpha,\beta)}(z) := \frac{(\alpha+1)_n}{n!} \, {}_2F_1\left(\begin{array}{c} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{array} ; \frac{1-z}{2} \right).$$

- Orthogonality relation:

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) (1-x)^\alpha (1+x)^\beta dx = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)n!} \delta_{m,n}.$$

- They satisfy the parity relation

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x).$$

Symmetric and antisymmetric Jacobi polynomials

The Gegenbauer function of the first kind is defined as

$$C_\lambda^\mu(z) := \frac{\sqrt{\pi}\,\Gamma(2\mu+\lambda)}{2^{2\mu-1}\Gamma(\mu)\Gamma(\mu+\frac{1}{2})\Gamma(\lambda+1)}\,{}_2F_1\left(\begin{matrix}-\lambda, 2\mu+\lambda\\ \mu+\frac{1}{2}\end{matrix};\frac{1-z}{2}\right),$$

and it is the clear extension of the Gegenbauer polynomial when the index is allowed to be a complex number as well as a non-negative integer.

The symmetric Jacobi polynomial is given by

$$P_n^{(\alpha,\alpha)}(x) = \frac{\Gamma(2\alpha+1)\Gamma(\alpha+1+n)}{\Gamma(\alpha+1)\Gamma(2\alpha+1+n)} C_n^{\alpha+\frac{1}{2}}(x).$$

The antisymmetric Jacobi polynomial is given by

$$P_n^{(\alpha,-\alpha)}(x) = \frac{\Gamma(2\alpha+1)\Gamma(1-\alpha+n)}{2^\alpha n!\Gamma(\alpha+1)}(1+x)^\alpha C_{n-\alpha}^{\alpha+\frac{1}{2}}(x)$$

Gegenbauer functions of the first and second kind

The Gegenbauer functions of the first and second kind (Durand) are related to associated Legendre functions of the first and second kind. The Gegenbauer function of the first kind is given by

$$C_\gamma^\mu(z)=\frac{\sqrt{\pi}}{2^{\mu-\frac12}}\frac{\Gamma(2\mu+\gamma)}{\Gamma(\mu)\Gamma(\gamma+1)}\frac{1}{(z^2-1)^{\frac{\mu}{2}-\frac14}}P_{\gamma+\mu-\frac12}^{\frac12-\mu}(z)$$

The Gegenbauer function of the second kind is given by

$$D_\gamma^\mu(z)=\frac{\mathrm{e}^{2\pi i(\mu-\frac14)}}{\sqrt{\pi}\,2^{\mu-\frac12}}\frac{\Gamma(2\mu+\gamma)}{\Gamma(\mu)\Gamma(\gamma+1)}\frac{1}{(z^2-1)^{\frac{\mu}{2}-\frac14}}Q_{\gamma+\mu-\frac12}^{\frac12-\mu}(z).$$

Associated Legendre functions are precisely those functions which satisfy a quadratic transformation of the Gauss hypergeometric function, which are very special.

Continuous q-Jacobi polynomials

The continuous q-Jacobi polynomials are defined in terms of Askey–Wilson polynomials. The Askey–Wilson polynomials are defined as

$$p_n(x;\mathbf{a}|q) := a^{-n}(ab,ac,ad;q)_n\, {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az^{\pm} \\ ab, ac, ad \end{matrix}; q, q\right),$$

where $x = \frac{1}{2}(z+z^{-1})$. The continuous q-Jacobi polynomials are defined as

$$P_n^{(\alpha,\beta)}(x|q) = \frac{a_1^n p_n(x;\mathbf{a}|q)}{(q,a_{13},a_{14};q)_n} = \frac{q^{(\frac{\alpha}{2}+\frac{1}{4})n}p_n(x;\mathbf{a}|q)}{(q,-q^{\frac{1}{2}(\alpha+\beta+1)},-q^{\frac{1}{2}(\alpha+\beta+2)};q)_n},$$

where

$$\mathbf{a} = \{a_1,a_2,a_3,a_4\} := \left\{q^{\frac{\alpha}{2}+\frac{1}{4}}, q^{\frac{\alpha}{2}+\frac{3}{4}}, -q^{\frac{\beta}{2}+\frac{1}{4}}, -q^{\frac{\beta}{2}+\frac{3}{4}}\right\}.$$

Alternate continuous q-Jacobi polynomials

Another q-analogue of the Jacobi polynomials is given by

$$P_n^{(\alpha,\beta)}(x;q) = \frac{(q^{\alpha+1},-q^{\beta+1};q)_n}{(\pm q;q)_m}\,{}_4\phi_3\left(\begin{matrix} q^{-n}, q^{\alpha+\beta+1+n}, q^{\frac12}z^{\pm} \\ q^{\alpha+1}, -q^{\beta+1}, -q \end{matrix};q,q\right)$$

$$= q^{-n\alpha}\frac{(-q^{\alpha+\beta+1};q)_n}{(-q;q)_n}P_n^{(\alpha,\beta)}(x|q^2).$$

This follows from the quadratic transformation (originally due to Singh)

$${}_4\phi_3\left(\begin{matrix} q^{-2n}, q^{2n}a^2, qb^2, c^2 \\ -a, -qa, q^2b^2c^2 \end{matrix};q^2,q^2\right) = \frac{(bc)^n(-q,-\frac{a}{bc};q)_n}{(-a,-qbc;q)_n}\,{}_4\phi_3\left(\begin{matrix} q^{-n}, q^n a, \frac{qb}{c}, \frac{c}{b} \\ -q, -\frac{a}{bc}, qbc \end{matrix};q,q\right)$$

and therefore has the following representation in terms of Askey–Wilson

$$P_n^{(\alpha,\beta)}(x;q) = \frac{q^{n/2}}{(q,-q,-q;q)_n}p_n(x;q^{\frac12},-q^{\frac12},q^{\alpha+1},-q^{\beta+1}|q).$$

Orthogonality and parity of continuous q-Jacobi polynomials

Orthogonality relation:

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x|q)P_n^{(\alpha,\beta)}(x|q)w_q^{(\alpha,\beta)}(x|q)\,\mathrm{d}x = h_n^{(\alpha,\beta)}(q)\delta_{m,n}.$$

$$w_q^{(\alpha,\beta)}(x|q) = \frac{(z^{\pm 2};q)_\infty(1-x^2)^{-1/2}}{(q^{\frac{\alpha}{2}+\frac{1}{4}}z^{\pm}, q^{\frac{\alpha}{2}+\frac{3}{4}}z^{\pm}, -q^{\frac{\beta}{2}+\frac{1}{4}}z^{\pm}, -q^{\frac{\beta}{2}+\frac{3}{4}}z^{\pm};q)_\infty},$$

$$h_n^{(\alpha,\beta)}(q) = \frac{2\pi\, q^{(\alpha+\frac{1}{2})n}(q^{\frac{\alpha+\beta}{2}+1}, q^{\frac{\alpha+\beta}{2}+\frac{3}{2}};q)_\infty(q^{\alpha+1},q^{\beta+1},q^{\frac{\alpha+\beta}{2}+\frac{1}{2}};q)_n}{(q,q^{\alpha+1},q^{\beta+1},-q^{\frac{\alpha+\beta}{2}+\frac{1}{2}},-q^{\frac{\alpha+\beta}{2}+1};q)_\infty(q,q^{\alpha+\beta+1},q^{\frac{\alpha+\beta}{2}+\frac{3}{2}};q)_n}.$$

They satisfy the parity relation

$$P_n^{(\alpha,\beta)}(-x|q) = (-1)^n q^{\frac{n}{2}(\alpha-\beta)}P_n^{(\beta,\alpha)}(x|q).$$

Motivation: The classical relation

The relation between the antisymmetric Jacobi function of the first kind and the Ferrers and Gegenbauer functions of the first kinds is

$$P_\gamma^{(\alpha,-\alpha)}(x) = \frac{\Gamma(\gamma+\alpha+1)}{\Gamma(\gamma+1)} \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}\alpha} \mathsf{P}_\gamma^{-\alpha}(x)$$

$$= \frac{\Gamma(2\alpha+1)\Gamma(\gamma-\alpha+1)}{2^\alpha\Gamma(\gamma+1)\Gamma(\alpha+1)}(1+x)^\alpha C_{\gamma-\alpha}^{\alpha+\frac{1}{2}}(x),$$

where $x \in \mathbb{C} \setminus ((-\infty,-1] \cup [1,\infty))$ and

$$P_\gamma^{(\alpha,-\alpha)}(z) = \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)} \left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\alpha} P_\gamma^{-\alpha}(z)$$

$$= \frac{\Gamma(2\alpha+1)\Gamma(\gamma-\alpha+1)}{2^\alpha\Gamma(\gamma+1)\Gamma(\alpha+1)}(z+1)^\alpha C_{\gamma-\alpha}^{\alpha+\frac{1}{2}}(z),$$

where $z \in \mathbb{C} \setminus (-\infty, 1]$.

The Plan

- In order to find a q-analogue of such a relation, we need to find q-analogue of Gegenbauer functions which will generalize continuous q-ultraspherical polynomials.
- Continuous q-ultraspherical polynomials and continuous q-Jacobi polynomials are specializations of Askey–Wilson polynomials.
- Hence, in order to find a q-analogue of Gegenbauer function, we will require continuous q-ultraspherical functions and therefore will need Askey–Wilson functions (of the first kind), the generalization of Askey–Wilson polynomials where the degree can be generalized to a complex number.
- The Askey–Wilson functions have been studied by other authors (Rahman, Groenevelt, Koelink, Stokman, Koornwinder, ...)

Continuous q-Jacobi and continuous q-ultraspherical functions from Askey–Wilson functions of the first kind

- continuous q-Jacobi functions

$$P_\lambda^{(\alpha,\beta)}(x|q) = \frac{q^{(\frac{1}{2}\alpha+\frac{1}{4})\lambda} p_\lambda(x; q^{\frac{\alpha}{2}+\frac{1}{4}}, q^{\frac{\alpha}{2}+\frac{3}{4}}, -q^{\frac{\beta}{2}+\frac{1}{4}}, -q^{\frac{\beta}{2}+\frac{3}{4}}|q)}{(q, -q^{\frac{\alpha+\beta}{2}+\frac{1}{2}}, -q^{\frac{\alpha+\beta}{2}+1}; q)_\lambda},$$

- continuous q-ultraspherical functions

$$C_\lambda(x; q^\gamma|q) = \frac{(q^{2\gamma}; q)_\lambda}{(q, -q^\gamma, \pm q^{\gamma+\frac{1}{2}}; q)_\lambda} p_\lambda(x; q^{\frac{\gamma}{2}}, q^{\frac{\gamma}{2}+\frac{1}{2}}, -q^{\frac{\gamma}{2}}, -q^{\frac{\gamma}{2}+\frac{1}{2}}|q)$$

- infinite q-shifted factorials

$$(a; q)_\lambda = \frac{(a; q)_\infty}{(q^\lambda a; q)_\infty}$$

The Askey–Wilson function of the first kind

- The Askey–Wilson function of the first kind is invariant under the transformation $z \mapsto z^{-1}$ as it should be.
- Consider the Askey–Wilson functions with parameters $\mathbf{a} := \{a, b_1, b_2, b_3\}$. The Askey–Wilson function of the first kind is symmetric under interchange of the variables b_1, b_2, b_3, but not under the interchange of the variable a unless $\lambda = n \in \mathbb{N}_0$ in which case it reduces to the Askey–Wilson polynomial.
- It is defined through basic hypergeometric representation as follows

$$p_\lambda(x;\mathbf{a}|q) := \frac{(\frac{q^{1-\lambda}}{az}, \left\{\frac{b_{123}}{b_s}\right\}, a^2; q)_\lambda}{a^\lambda(\frac{q^{1-\lambda}}{a^2}, \frac{b_{123}}{z}; q)_\lambda}$$

$$\times\, {}_8W_7\left(\frac{b_{123}}{qz}; q^{-\lambda}, q^{\lambda-1}ab_{123}, \frac{b_1}{z}, \frac{b_2}{z}, \frac{b_3}{z}; q, \frac{qz}{a}\right),$$

where $|qz| < |a|$.

Askey–Wilson function of the first kind

$$
\begin{aligned}
&p_\lambda(x;\mathbf{a}|q)\\
&=\frac{\vartheta(a^2;q)(cd,bz,\frac{qc}{a},\frac{qd}{a},q^\lambda\frac{bcd}{z},\frac{q^{1-\lambda}}{az};q)_\infty}{a^\lambda\vartheta(q^\lambda a^2;q)(\frac{q}{a}z^\pm,\frac{qcd}{az},q^\lambda bc,q^\lambda bd,q^\lambda cd;q)_\infty}\,{}_8W_7\left(\frac{cd}{az};\frac{q}{az},\frac{c}{z},\frac{d}{z},q^\lambda cd,\frac{q^{1-\lambda}}{ab};q,bz\right)\\
&=\frac{\vartheta(a^2;q)(bc,bd,cd,\frac{q^{1-\lambda}}{az},q^\lambda\frac{bcd}{z};q)_\infty}{a^\lambda\vartheta(q^\lambda a^2;q)(\frac{bcd}{z},\frac{q}{az},q^\lambda bc,q^\lambda bd,q^\lambda cd;q)_\infty}\,{}_8W_7\left(\frac{bcd}{qz};q^{-\lambda},q^{\lambda-1}abcd,\frac{b}{z},\frac{c}{z},\frac{d}{z};q,\frac{qz}{a}\right)\\
&=\frac{\vartheta(a^2;q)(bd,cd,\frac{qd}{a},\frac{q^{1-\lambda}}{a}z^\pm;q)_\infty}{a^\lambda\vartheta(q^\lambda a^2;q)(\frac{q}{a}z^\pm,q^\lambda bd,q^\lambda cd,\frac{q^{1-\lambda}d}{a};q)_\infty}\,{}_8W_7\left(\frac{q^{-\lambda}d}{a};q^{-\lambda},\frac{q^{1-\lambda}}{ab},\frac{q^{1-\lambda}}{ac},dz^\pm;q,q^\lambda bc\right)\\
&=\frac{\vartheta(a^2;q)(\frac{qb}{a},\frac{qc}{a},\frac{qd}{a},q^{\lambda-1}abcd,\frac{q^{1-\lambda}}{a}z^\pm;q)_\infty}{a^\lambda\vartheta(q^\lambda a^2;q)(\frac{q}{a}z^\pm,q^\lambda bc,q^\lambda bd,q^\lambda cd,\frac{q^{2-\lambda}}{a^2};q)_\infty}\\
&\qquad\times{}_8W_7\left(\frac{q^{1-\lambda}}{a^2};\frac{q}{a}z^\pm,\frac{q^{1-\lambda}}{ab},\frac{q^{1-\lambda}}{ac},\frac{q^{1-\lambda}}{ad};q,q^{\lambda-1}abcd\right),
\end{aligned}
$$

where in order for the nonterminating ${}_8W_7$ to be convergent, one requires respectively $|bz|<1$, $|qz|<|a|$, $|q^\lambda bc|<1$, $|q^{1-\lambda}|<|ac|$, $|q^{\lambda-1}abcd|<1$, and we assume there are no vanishing denominator factors.

Analytic continuation in λ and the correct choice for a

$$p_\lambda(x;\mathbf{a}|q)=\frac{\vartheta(a^2;q)(q^{-\lambda},\frac{qb}{a},\frac{qc}{a},\frac{qd}{a},q^\lambda bcdz^{\pm};q)_\infty}{a^\lambda\vartheta(q^\lambda a^2;q)(\frac{q}{a}z^{\pm},q^\lambda bc,q^\lambda bd,q^\lambda cd,q^{\lambda+1}\frac{bcd}{a},\frac{q}{a}z^{\pm};q)_\infty}\times{}_8W_7\left(q^\lambda\frac{bcd}{a};q^\lambda bc,q^\lambda bd,q^\lambda cd,\frac{q}{a}z^{\pm};q,q^{-\lambda}\right)$$

$$=\frac{\vartheta(a^2;q)(bc,cd,\frac{qc}{a},q^\lambda bcdz^{\pm},\frac{q^{1-\lambda}}{ac};q)_\infty}{a^\lambda\vartheta(q^\lambda a^2;q)(\frac{q}{a}z^{\pm},q^\lambda bc,q^\lambda bd,q^\lambda cd,q^\lambda bc^2d;q)_\infty}\times{}_8W_7\left(q^{\lambda-1}bc^2d;q^{\lambda-1}abcd,cz^{\pm},q^\lambda bc,q^\lambda cd;q,\frac{q^{1-\lambda}}{ac}\right).$$

Similarly for continuous q-Jacobi functions and also for continuous q-ultraspherical functions.

Continuous q-Jacobi functions of the first kind

- We are able to get good q-analogues of the Jacobi function of the first kind, namely by choosing a equal to either $a_3 = -q^{\frac{\beta}{2}+\frac{1}{4}}$ or $a_4 = -q^{\frac{\beta}{2}+\frac{3}{4}}$ in the Askey–Wilson function. One finds good q-analogues of the Jacobi function of the first kind in the respect that their $q \to 1^-$ limit approaches the Jacobi function of the first kind with the correct hypergeometric behavior.

$$P_n^{(\alpha,\beta)}(-x|q) = (-1)^n q^{\frac{n}{2}(\alpha-\beta)} P_n^{(\beta,\alpha)}(x|q).$$

$$P_\gamma^{(\alpha,\beta)}(-z) = \frac{\sin(\pi(\beta+\gamma))}{\sin(\pi\beta)} P_\gamma^{(\beta,\alpha)}(z) - \frac{\sin(\pi\gamma)}{\sin(\pi\beta)} \frac{\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{\Gamma(\alpha+\beta+\gamma+1)\Gamma(\gamma+1)} \left(\frac{2}{1-z}\right)^\beta P_{\gamma+\beta}^{(-\beta,\alpha)}(z).$$

- We will call these continuous q-Jacobi functions of type I and with the choice $a = a_4$, we will call these continuous q-Jacobi functions of type II.

Continuous q-Jacobi functions of the first kind

$$P_\lambda^{(\alpha,\beta)}(x|q) = \left(-q^{\frac{\alpha-\beta}{2}}\right)^\lambda \frac{\vartheta(q^{\beta+\frac12};q)}{\vartheta(q^{\beta+\frac12+\lambda};q)(q;q)_\infty} \mathsf{W}_k^{(\alpha,\beta)}(\lambda;z|q),$$

where $\mathsf{W}_k := \mathsf{W}_k^{(\alpha,\beta)}(\lambda;z|q)$, with $|q^{\frac{\beta}{2}+\frac34}z|,\ |q^{-\frac{\beta}{2}+\frac34}z|,\ |q^{\frac{\alpha+\beta+2}{2}+\lambda}|,\ |q^{\alpha+\beta+1+\lambda}|<1,$

$$\mathsf{W}_1^{(\alpha,\beta)} := \frac{(q^{\alpha+1}, -q^{\frac{2+\alpha-\beta}{2}}, -q^{\frac{3+\alpha-\beta}{2}}, q^{\lambda+1}, -q^{\frac{\alpha+\beta+1}{2}+\lambda}, -q^{\frac34+\frac{\beta}{2}}z, -q^{\frac34-\frac{\beta}{2}-\lambda}z^{-1}, -q^{\frac74+\alpha+\frac{\beta}{2}+\lambda}z^{-1};q)_\infty}{(-q^{\frac{\alpha+\beta+1}{2}}, -q^{\frac{\alpha+\beta+2}{2}}, q^{\alpha+1+\lambda}, -q^{\frac{\alpha+\beta+3}{2}+\lambda}, -q^{\frac34-\frac{\beta}{2}}z^{\pm}, -q^{\frac74+\alpha-\frac{\beta}{2}}z^{-1};q)_\infty}$$
$$\times {}_8W_7\left(-q^{\frac34+\alpha-\frac{\beta}{2}}z^{-1}; q^{-\beta-\lambda}, q^{\alpha+1+\lambda}, q^{\frac14+\frac{\alpha}{2}}z^{-1}, q^{\frac34+\frac{\alpha}{2}}z^{-1}, -q^{\frac34-\frac{\beta}{2}}z^{-1}; q, -q^{\frac34+\frac{\beta}{2}}z\right)$$

$$\mathsf{W}_2^{(\alpha,\beta)} := \frac{(q^{\alpha+1}, -q^{\frac{\alpha+\beta+3}{2}}, q^{\lambda+1}, -q^{\frac{\alpha+\beta+1}{2}+\lambda}, -q^{\frac34-\frac{\beta}{2}-\lambda}z^{-1}, -q^{\frac74+\alpha+\frac{\beta}{2}+\lambda}z^{-1};q)_\infty}{(-q^{\frac{\alpha+\beta+1}{2}}, q^{\alpha+1+\lambda}, -q^{\frac{\alpha+\beta+3}{2}+\lambda}, -q^{\frac34-\frac{\beta}{2}}z^{-1}, -q^{\frac74+\alpha+\frac{\beta}{2}}z^{-1};q)_\infty}$$
$$\times {}_8W_7\left(-q^{\frac34+\alpha+\frac{\beta}{2}}z^{-1}; q^{-\lambda}, q^{\alpha+\beta+1+\lambda}, -q^{\frac34+\frac{\beta}{2}}z^{-1}, q^{\frac14+\frac{\alpha}{2}}z^{-1}, q^{\frac34+\frac{\alpha}{2}}z^{-1}; q, -q^{\frac34-\frac{\beta}{2}}z\right),$$

$$\mathsf{W}_3^{(\alpha,\beta)} := \frac{(q^{\alpha+1}, -q^{\frac{\alpha+\beta+3}{2}}, -q^{\frac{3+\alpha-\beta}{2}}, q^{\lambda+1}, -q^{\frac{\alpha+\beta+1}{2}+\lambda}, -q^{\frac{\alpha+\beta+2}{2}+\lambda}, -q^{\frac34-\frac{\beta}{2}-\lambda}z^{\pm};q)_\infty}{(-q^{\frac{\alpha+\beta+1}{2}}, -q^{\frac{\alpha+\beta+2}{2}}, q^{\alpha+1+\lambda}, -q^{\frac{\alpha+\beta+3}{2}+\lambda}, -q^{\frac{3+\alpha-\beta}{2}-\lambda}, -q^{\frac34-\frac{\beta}{2}}z^{\pm};q)_\infty}$$
$$\times {}_8W_7\left(-q^{\frac{\alpha-\beta+1}{2}-\lambda}; q^{-\lambda}, q^{-\beta-\lambda}, -q^{\frac{1-\beta-\alpha}{2}-\lambda}, q^{\frac34+\frac{\alpha}{2}}z^{\pm}; q, -q^{\frac{\alpha+\beta+2}{2}+\lambda}\right),$$

$$\mathsf{W}_4^{(\alpha,\beta)} := \frac{(q^{\frac32}, -q^{\frac{2+\alpha-\beta}{2}}, -q^{\frac{3+\alpha-\beta}{2}}, q^{\lambda+1}, q^{\alpha+\beta+1+\lambda}, -q^{\frac{\alpha+\beta+1}{2}+\lambda}, -q^{\frac34-\frac{\beta}{2}-\lambda}z^{\pm};q)_\infty}{(-q^{\frac{\alpha+\beta+1}{2}}, -q^{\frac{\alpha+\beta+2}{2}}, q^{\alpha+1+\lambda}, q^{\frac32-\beta-\lambda}, -q^{\frac{\alpha+\beta+3}{2}+\lambda}, -q^{\frac34-\frac{\beta}{2}}z^{\pm};q)_\infty}$$
$$\times {}_8W_7\left(q^{\frac12-\beta-\lambda}; q^{-\beta-\lambda}, -q^{\frac{-\alpha-\beta}{2}-\lambda}, -q^{\frac{-\alpha-\beta+1}{2}-\lambda}, -q^{\frac34-\frac{\beta}{2}}z^{\pm}; q, q^{\alpha+\beta+1+\lambda}\right).$$

Continuous q-ultraspherical functions of the first kind

$$C_\lambda(x;q^\gamma|q)=q^{-\frac{\gamma\lambda}{2}}\frac{\vartheta(q^\gamma;q)(q^\gamma;q)_\infty}{\vartheta(q^{\gamma+\lambda};q)(q;q)_\infty}\mathsf{C}_k(\lambda;z|q)$$

where $\mathsf{C}_k:=\mathsf{C}_k(\lambda;z;q^\gamma|q)$, with $|q^{1-\frac{\gamma}{2}}z|,\ |q^{\gamma+\frac12+\lambda}|,\ |q^{\frac{\gamma+1}{2}}z|,\ |q^{2\gamma+\lambda}|<1$,

$$\mathsf{C}_1:=\frac{(-q^{\gamma+1},\pm q^{\gamma+\frac12},q^{\lambda+1},-q^{\gamma+\lambda},q^{1-\frac{\gamma}{2}-\lambda}z^{-1},q^{1+\frac{3\gamma}{2}+\lambda}z^{-1};q)_\infty}{(q^{2\gamma+\lambda},-q^{\gamma+\lambda+1},q^{1-\frac{\gamma}{2}}z^{-1},q^{1+\frac{3\gamma}{2}}z^{-1};q)_\infty}$$
$$\times\,{}_8W_7\left(q^{\frac{3\gamma}{2}}z^{-1};q^{-\lambda},q^{2\gamma+\lambda},-q^{\frac{\gamma}{2}}z^{-1},\pm q^{\frac{\gamma+1}{2}}z^{-1};q,q^{1-\frac{\gamma}{2}}z\right),$$

$$\mathsf{C}_2:=\frac{(-q^{\frac32},-q^{\gamma+1},q^{\gamma+\frac12},q^{\lambda+1},-q^{\gamma+\lambda},-q^{\gamma+\frac12+\lambda},q^{1-\frac{\gamma}{2}-\lambda}z^{\pm};q)_\infty}{(-q^{\frac32-\lambda},-q^{\gamma+1+\lambda},q^{2\gamma+\lambda},q^{1-\frac{\gamma}{2}}z^{\pm};q)_\infty}$$
$$\times\,{}_8W_7\left(-q^{\frac12-\lambda};q^{-\lambda},-q^{1-\gamma-\lambda},q^{\frac12-\gamma-\lambda},-q^{\frac{\gamma+1}{2}}z^{\pm};q,-q^{\gamma+\frac12+\lambda}\right),$$

$$\mathsf{C}_3:=\frac{(-q,q^{\frac32},q^{\gamma+\frac12},q^{\lambda+1},-q^{\gamma+\lambda},q^{\frac{\gamma+1}{2}}z,q^{1-\frac{\gamma}{2}-\lambda}z^{-1},q^{1+\frac{3\gamma}{2}+\lambda}z^{-1};q)_\infty}{(q^{2\gamma+\lambda},-q^{\gamma+1+\lambda},q^{1-\frac{\gamma}{2}}z^{\pm},q^{\frac{3+\gamma}{2}}z^{-1};q)_\infty}$$
$$\times\,{}_8W_7\left(q^{\frac{\gamma+1}{2}}z^{-1};q^{\frac12+\gamma+\lambda},q^{\frac12-\gamma-\lambda},-q^{\frac{\gamma}{2}}z^{-1},q^{1-\frac{\gamma}{2}}z^{-1},-q^{\frac{\gamma+1}{2}z^{-1}};q,q^{\frac{\gamma+1}{2}}z\right),$$

$$\mathsf{C}_4:=\frac{(-q,\pm q^{\frac32},q^{\gamma+\frac12},q^{\lambda+1},-q^{\gamma+\lambda},q^{1-\frac{\gamma}{2}-\lambda}z^{\pm};q)_\infty}{(q^{\gamma+\frac12},-q^{\gamma+1+\lambda},q^{2-\gamma-\lambda},q^{1-\frac{\gamma}{2}}z^{\pm};q)_\infty}$$
$$\times\,{}_8W_7\left(q^{1-\gamma-\lambda};\pm q^{\frac12-\gamma-\lambda},-q^{1-\gamma-\lambda},q^{1-\frac{\gamma}{2}}z^{\pm};q,q^{2\gamma+\lambda}\right).$$

Wilson function of the first kind

The Wilson polynomials are symmetric in four parameters a, b, c, d. Define $\mathbf{a} := \{a, b, c, d\}$. Then the Wilson polynomials are defined as

$$W_n(x^2;\mathbf{a}) := (a+b, a+c, a+d)_n \, {}_4F_3\left(\begin{matrix} -n, n+a+b+c+d-1, a\pm ix \\ a+b, a+c, a+d \end{matrix};1\right),$$

We take as our standard limit transition the relation

$$W_\lambda(x^2;\mathbf{a}) = \lim_{q\to 1^-} (1-q)^{-3\lambda}\, p_\lambda(\tfrac{1}{2}(q^{ix}+q^{-ix}); q^{\mathbf{a}}|q),$$

$(a, b, c, d) \mapsto (q^a, q^b, q^c, q^d)$, $z = q^{ix}$, which is akin to the limit transition from Askey–Wilson polynomials to Wilson polynomials.

$$W(a;b,c,d,e,f) := {}_7F_6\left(\begin{matrix} a, \frac{a}{2}+1, b, c, d, e, f \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a-f \end{matrix};1\right)$$

which is absolutely convergent if $\Re(2a-(b+c+d+e+f)+2) > 0$.

Wilson function of the first kind II

$$W_\lambda(x^2;\mathbf{a}) = \frac{\Gamma(2a+\lambda, 1-2a-\lambda, 1-a\pm ix, c+d-a+1-ix, b+c+\lambda, b+d+\lambda, c+d+\lambda)}{\Gamma(2a, 1-2a, c+d, b+ix, c-a+1, d-a+1, b+c+d+\lambda-ix, 1-a-\lambda-ix)}$$
$$\times W(c+d-a-ix; 1-a-ix, c-ix, d-ix, c+d+\lambda, 1-\lambda-a-b),$$
$$= \frac{\Gamma(2a+\lambda, 1-\lambda-2a, b+d+\lambda, c+d+\lambda, d-a+1-\lambda, 1-a\pm ix)}{\Gamma(2a, 1-2a, b+d, c+d, 1+d-a, 1-a-\lambda\pm ix)}$$
$$\times W(d-a-\lambda; -\lambda, 1-a-b-\lambda, 1-a-c-\lambda, d\pm ix),$$
$$= \frac{\Gamma(2a+\lambda, 1-\lambda-2a, b+c+d-ix, 1-a-ix, b+c+\lambda, b+d+\lambda, c+d+\lambda)}{\Gamma(2a, 1-2a, b+c, b+d, c+d, 1-a-\lambda-ix, b+c+d+\lambda-ix)}$$
$$\times W(b+c+d-1-ix; -\lambda, \lambda+a+b+c+d-1, b-ix, c-ix, d-ix),$$
$$= \frac{\Gamma(2a+\lambda, 1-\lambda-2a, 1-a\pm ix, b+c+\lambda, b+d+\lambda, c+d+\lambda, 2-\lambda-2a)}{\Gamma(2a, 1-2a, b-a+1, c-a+1, d-a+1, a+b+c+d+\lambda-1, 1-\lambda-a\pm ix)}$$
$$\times W(1-\lambda-2a; 1-a\pm ix, 1-\lambda-a-b, 1-\lambda-a-c, 1-\lambda-a-d),$$

which are absolutely convergent if
$\Re(b+ix, b+c+\lambda, 1-a+ix, a+b+c+d-1+\lambda) > 0.$

Wilson polynomials

New representations for the Wilson polynomials

$$W_n(x^2;\mathbf{a}) = (a+ix,b+ix,c+d)_n\ {}_4F_3\left(\begin{matrix}-n,1-a-b-n,c-ix,d-ix\\ 1-a-ix-n,1-b-ix-n,c+d\end{matrix};1\right)$$

$$= (-1)^n\frac{(a\pm ix)_n(a+b+c+d-1)_{2n}}{(a+b+c+d-1)_n}$$

$$\times\ {}_4F_3\left(\begin{matrix}-n,1-a-b-n,1-a-c-n,1-a-d-n\\ 1-a\pm ix-n,2-a-b-c-d-2n\end{matrix};1\right).$$

Antisymmetry relation

- Obtained antisymmetry relation for continuous q-Jacobi polynomials/functions

$$P_n^{(\alpha,-\alpha)}(x) = \frac{\Gamma(2\alpha+1)\Gamma(1-\alpha+n)}{2^\alpha n!\Gamma(\alpha+1)}(1+x)^\alpha C_{n-\alpha}^{\alpha+\frac{1}{2}}(x)$$

Result

$$P_\lambda^{(\alpha,-\alpha)}(x|q) = \mathrm{e}^{i\pi(2\lambda-\alpha)} q^{\alpha\lambda} \left(q^{\frac{\alpha}{2}+\frac{1}{4}}\right)^{\lambda-\alpha}$$

$$\times \frac{(q^{\frac{1}{2}}, q^{\lambda+1}, q^{\frac{1}{2}\pm\lambda}, -q^{\frac{3}{4}-\frac{\alpha}{2}}z^{\pm}; q)_\infty}{(q^{\alpha+\frac{1}{2}}, q^{1+\lambda-\alpha}, q^{\frac{1}{2}+\lambda\pm\alpha}, -q^{\frac{\alpha}{2}+\frac{3}{4}}z^{\pm}; q)_\infty} C_{\lambda-\alpha}(x; q^{\alpha+\frac{1}{2}}|q).$$

$$\lim_{q\to 1^-} P_\lambda^{(\alpha,\beta)}(x|q) = P_\lambda^{(\alpha,\beta)}(x), \quad \lim_{q\to 1^-} C_\lambda(x; q^\mu|q) = \frac{\mathrm{e}^{-i\pi\lambda}\sin(\pi\mu)}{\sin(\pi(\mu+\lambda))} C_\lambda^\mu(x)$$

Future directions of work

- Robert S. Maier, *Associated Legendre Functions and Spherical Harmonics of Fractional Degree and Order*, Constr. Approx. 48:235-281, 2018. Several (of many) examples of *Dihedral monodromy* Legendre (and Ferrers) functions

$$P^{\alpha}_{-\frac{1}{2}\pm(n+\frac{1}{2})}(z) = \frac{n!}{\Gamma(n-\alpha+1)}\left(\frac{z+1}{z-1}\right)^{\frac{\alpha}{2}} P_n^{(-\alpha,\alpha)}(z).$$

$$Q^{\frac{1}{2}+m}_{-\frac{1}{2}+\alpha}(z) = \frac{i(-1)^m\sqrt{\pi/2}}{(z^2-1)^{\frac{1}{4}}(z+\sqrt{z^2-1})^{\alpha}} P_m^{(\alpha,-\alpha)}\left(\frac{z}{\sqrt{z^2-1}}\right)$$

$$Q^{-\frac{1}{2}-m}_{-\frac{1}{2}+\alpha}(z) = \frac{-i\sqrt{\pi/2}}{\alpha(1-\alpha)_m(1+\alpha)_m} \times(z^2-1)^{-\frac{1}{4}}(z+\sqrt{z^2-1})^{-\alpha} P_m^{(\alpha,-\alpha)}\left(\frac{z}{\sqrt{z^2-1}}\right)$$

Future directions of work

- Work with Rahman's Askey–Wilson function of the second kind to obtain corresponding formulas of antisymmetry relations for continuous q-Jacobi functions of the second kind in terms of continuous q-ultraspherical functions of the second kind.
- Treat quadratic continuous q-Jacobi polynomials
- q-functions which generalize orthogonal polynomials in the q-Askey-scheme are a rich source of exploration