# Padé Approximants for Functions with Four Branch Points.

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# Diagonal Padé approximants.

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$$f(z) = \sum_{i=0}^{\infty} \frac{\mu_i}{z^i}.$$

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**The nth diagonal Padé approximant,**

$$[n/n]_f(z) := \frac{P_n(z)}{Q_n(z)}, \quad \deg Q_n \le n, \ \deg P_n \le n,$$

is a rational function whose Laurent series agrees with the Laurent series of $f$ "as far as possible."

More precisely, the approximation property is that the **linearized error function** satisfies

$$R_n(z) := (Q_n f - P_n)(z) = \mathcal{O}(z^{-n-1}) \quad \text{as} \quad z \to \infty.$$

It follows that

$$f(z) - [n/n]_f(z) = \frac{R_n(z)}{Q_n(z)}.$$

**Remark**

The pair $(Q_n, P_n)$ may not be unique, but the ratio $[n/n]_f$ is.

# Stahl class.

Suppose $f$ has an analytic continuation originating from infinity to $\mathbb{C} \setminus E_f$ so that

- logarithmic capacity of $E_f$ is zero, and
- $f$ has two distinct analytic continuations to some point in $\mathbb{C} \setminus E_f$.

**Theorem** (H. Stahl)
There exists a "minimal" compact set $\Delta_f$ so that for $z \in \mathbb{C} \setminus \Delta_f$,

$$[n/n]_f(z) \to f(z) \quad \text{in capacity.}$$

# Stahl's work.

- H. Stahl, **Extremal domains associated with an analytic function. I, II.** *Complex Variables Theory Appl.*, vol. 4, pp. 311-324, 325-338, 1985.
- H. Stahl, **Structure of extremal domains associated with an analytic function.** *Complex Variables Theory Appl.*, vol. 4, pp. 339-356, 1985.
- H. Stahl, **Orthogonal polynomials with complex valued weight function. I, II.** *Constr. Approx.*, vol. 2, no. 3, pp. 225-240, 241-251, 1986.
- H. Stahl, **The convergence of Padé approximants to functions with branch points**, *Journal of Approximation Theory*, vol. 91, no. 2, pp. 139-204, 1997.

# What's $\Delta_f$?

Let's stare at some pictures to get a sense of what the set $\Delta_f$ looks like.

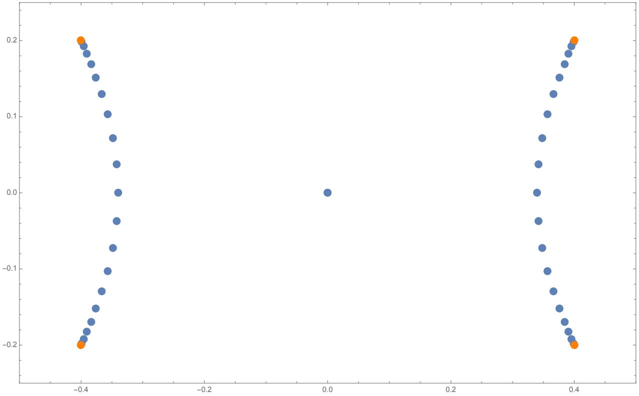


Figure: Poles of $[40/40]_{f_1}$ with $f_1(z) = \sqrt{25 - \frac{6}{z^2} + \frac{1}{z^4}}$

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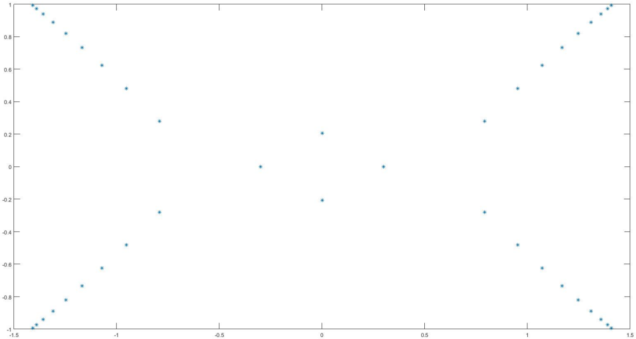


Figure: Poles of $[40/40]_{f_2}$ with $f_2(z) = \sqrt[4]{1 - \frac{2}{z^2} + \frac{9}{z^4}}$

# What's $\Delta_f$?

Let's stare at some pictures to get a sense of what the set $\Delta_f$ looks like.

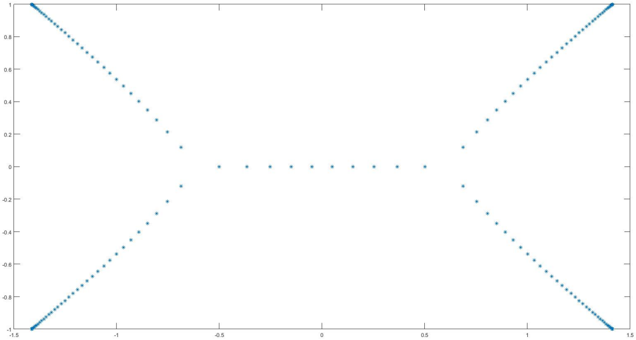


Figure: Poles of $[150/150]_{f_2}$ with $f_2(z) = \sqrt[4]{1 - \frac{2}{z^2} + \frac{9}{z^4}}$

# Orthogonal polynomials.

We now further specialize to functions $f(z)$ of the form

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\Delta} \frac{\rho(s)\mathrm{d}s}{s - z}.$$

One can now check that the denominator of $[n/n]_f$ is a **non-Hermitian orthogonal polynomial.**

$$\int_{\Delta} z^k Q_n(z)\rho(z)\mathrm{d}z = 0 \quad \text{for} \quad k = 0, 1, \ldots, n - 1.$$

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$$\int_{\Delta} z^k Q_n(z)\rho(z)\mathrm{d}z = 0 \quad \text{for} \quad k = 0, 1, \ldots, n-1.$$

**Remark**

We observed before that $Q_n$ may not be unique. There is, however, a unique *minimal-degree* denominator. From here on, we denote by $Q_n$ this minimal-degree polynomial.

# Convergence results.

When $f$ only has branch point $\{a_1, a_2, \ldots, a_k\}$ of *algebraic or logarithmic type* that are in generic position,

**Theorem** (A. Aptekarev & M. Yattselev)[1]
For any $\varepsilon > 0$, there is $\mathbb{N}_\varepsilon \subseteq \mathbb{N}$ so that as $n \to \infty$, $n \in \mathbb{N}_\varepsilon$,

$$Q_n(z) = \gamma_n \left(1 + \mathcal{O}_\varepsilon\left(n^{-1}\right)\right) \Psi_n^{(0)}(z) + \gamma_n \mathcal{O}_\varepsilon\left(n^{-1}\right) \Psi_{n-1}^{(0)}(z),$$

$$(wR_n)(z) = \gamma_n \left(1 + \mathcal{O}_\varepsilon\left(n^{-1}\right)\right) \Psi_n^{(1)}(z) + \gamma_n \mathcal{O}_\varepsilon\left(n^{-1}\right) \Psi_{n-1}^{(1)}(z),$$

where $w^2(z) = \prod_j (z - e_j)$, where $\{e_j\}$ points on $\Delta$ with odd valence.

[1] Padé approximants for functions with branch points—strong asymptotics of Nuttall-Stahl polynomials. *Acta Math.* 215 (2015), no. 2, 217–280.

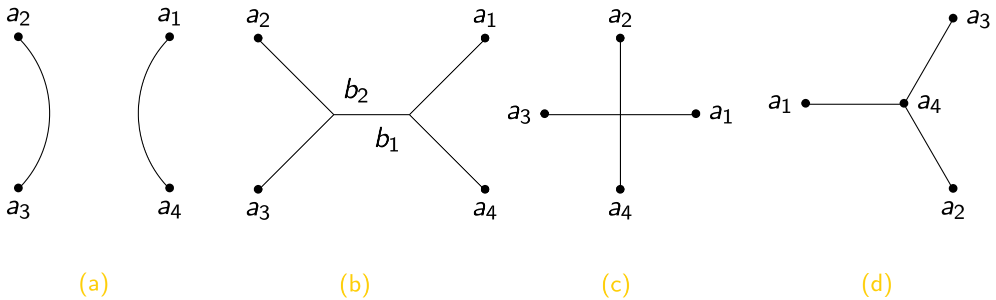



Figure: Possible non-colinear arrangements of branch points $a_i$'s.

# The Formula.

$$Q_n(z) = \gamma_n \left(1 + \mathcal{O}_\varepsilon\left(n^{-1}\right)\right) \Psi_n^{(0)}(z) + \gamma_n \mathcal{O}_\varepsilon\left(n^{-1}\right) \Psi_{n-1}^{(0)}(z), \quad \text{and}$$

$$f(z) - [n/n]_f = \frac{1}{w(z)} \cdot \frac{\left(1 + \mathcal{O}_\varepsilon\left(n^{-1}\right)\right) \Psi_n^{(1)}(z) + \mathcal{O}_\varepsilon\left(n^{-1}\right) \Psi_{n-1}^{(1)}(z)}{\left(1 + \mathcal{O}_\varepsilon\left(n^{-1}\right)\right) \Psi_n^{(0)}(z) + \mathcal{O}_\varepsilon\left(n^{-1}\right) \Psi_{n-1}^{(0)}(z)},$$

as $\mathbb{N}_\varepsilon \ni n \to \infty$.

In (a), the worst we can do is $\mathbb{N}_\varepsilon = 2\mathbb{N}$ or $\mathbb{N}_\varepsilon = \mathbb{N} \setminus 2\mathbb{N}$.
Functions $\Psi^{(i)}$ are known as *Baker-Akhiezer* functions;

- $\Psi_n^{(0)} \sim z^n$ as $z \to \infty$ and is geometrically large on closed subsets of $\mathbb{C} \setminus \Delta$ while $\Psi_n^{(1)}$ is geometrically small there,
- $\Psi_n^{(j)}$ can have at most one simple zero in $\mathbb{C}$.

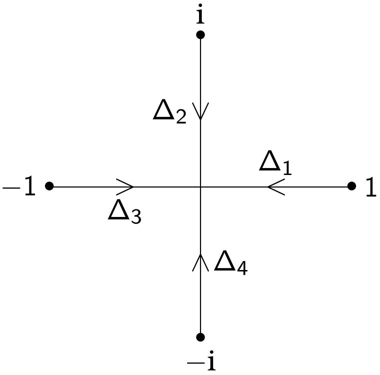
# Singular example.

Consider the OPs corresponding to the weight

$$\rho(z) = \frac{-\mathrm{i}^{4-j}}{|(z^4-1)^{1/4}|}, \quad z \in \Delta_j$$

for an appropriately chosen branch.
Then,

$$Q_{4n}(z) = P_{n,1}(z^4),$$
$$Q_{4n+1}(z) = Q_{4n+2}(z) = Q_{4n+3}(z) = zP_{n,2}(z^4)$$




where $P_{n,1}, P_{n,2}$ are orthogonal on $[0,1]$ with respect to $z^{-3/4}(1-z)^{-1/4}$ and $z^{1/4}(1-z)^{-1/4}$, respectively.

# Singular example.

**Theorem** (A.B. & M. Yattselev)[2]
For $\varepsilon > 0$ small enough, there is $\mathbb{N}_\varepsilon \subseteq \mathbb{N}$ so that as $n \to \infty$, $n \in \mathbb{N}_\varepsilon$,

$$Q_n(z) = \gamma_n \left(1 + \frac{L_{n,1}}{z} + o_\varepsilon(1)\right) \Psi_n^{(0)}(z) + \gamma_n \left(\frac{L_{n,2}}{z} + o_\varepsilon(1)\right) \Psi_{n-1}^{(0)}(z)$$

Furthermore, $\mathbb{N}_\varepsilon$ has "gaps" of lengths at most 2.

Estimates for the $o_\varepsilon(1)$ terms are available and depend on the weight of orthogonality.

[2] Asymptotics of polynomials orthogonal on a cross with a Jacobi-type weight. *Complex Analysis and Operator Theory* (2020) 14:9.

# The Final Frontier.

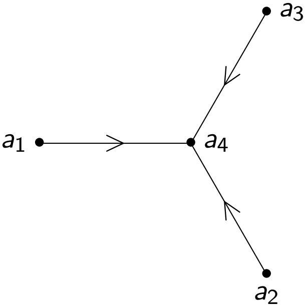
Let $\hat{\rho}(z)$ be an analytic function in a neighborhood of $[-1,0]$ and define

$$\rho(z) = \hat{\rho}(z^3), \quad z \in \Delta.$$

Then, polynomials $Q_n$ can be written in terms of the polynomials orthogonal on $[-1,0]$, $\hat{Q}_n$,

$$Q_{3n}(z) = Q_{3n+1}(z) = Q_{3n+2}(z) = \hat{Q}_n(z^3).$$

It follows from classical formulas that $Q_n$ do not satisfy the usual asymptotic formula for any $n$.



# It's not all bad.

Let $\varrho_i(a_0) = \lim_{z \to a_0} \rho_i(z)(z - a_i)^{-\alpha_i}$ and define

$$b_1(\rho) := -\frac{e^{\pi \mathrm{i} \alpha_0} \varrho_2(a_0) + e^{-\pi \mathrm{i} \alpha_0} \varrho_3(a_0)}{\varrho_1(a_0)},$$

$$b_2(\rho) := -\frac{\varrho_1(a_0) + \varrho_3(a_0)}{e^{\pi \mathrm{i}} \varrho_2(a_0)}, \qquad b_3(\rho) := -\frac{\varrho_1(a_0) + \varrho_2(a_0)}{e^{-\pi \mathrm{i}} \varrho_3(a_0)}$$

One can immediately check that

$$b_1 + b_2 + b_3 - b_1 b_2 b_3 = 2\cos(\pi \alpha_0).$$

# It's not all bad.

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We say a weight $\rho \in W^{\mathrm{sing}}$ if $(b_1(\rho), b_2(\rho), b_3(\rho))$ are Stokes parameters of a solution Painlevé XXXIV,

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 4u^2(x) + 2xu(x) + \frac{1}{2u(x)}\left(\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - \alpha_0^2\right).$$

with a pole at $x = 0$. Otherwise, we say $\rho \in W^{\mathrm{reg}}$

# It's not all bad.

**Theorem** (A.B. & M. Yattselev)[3] For $\varepsilon > 0$ small enough and $\rho \in W^{\mathrm{reg}}$, there exists $\mathbb{N}_\varepsilon \subseteq \mathbb{N}$ so that as $n \to \infty, n \in \mathbb{N}_\varepsilon$,

$$Q_n(z) = \gamma_n \left(1 + \mathcal{O}_\varepsilon\left(n^{-1/3}\right)\right) \Psi_n^{(0)}(z) + \gamma_n \left(\mathcal{O}_\varepsilon\left(n^{-1/3}\right)\right) \Psi_{n-1}^{(0)}(z)$$

When, on the other hand, $\rho \in W^{\mathrm{sing}}$, there is a subsequence $\mathbb{N}_\varepsilon^{\mathrm{sing}} \subseteq \mathbb{N}_\varepsilon$ so that as $n \to \infty, n \in \mathbb{N}_\varepsilon^{\mathrm{sing}}$,

$$Q_n(z) = \gamma_n \left(1 + \frac{L_{n,1}}{z - a_4} + \mathcal{O}_\varepsilon\left(n^{-1/3}\right)\right) \Psi_n^{(0)}(z) + \gamma_n \left(\frac{L_{n,2}}{z - a_4} + \mathcal{O}_\varepsilon\left(n^{-1/3}\right)\right) \Psi_{n-1}^{(0)}(z)$$

[3] Non-Hermitian orthogonal polynomials on a trefoil. *Preprint.*

# The end.

**Thank you for listening!**

Questions?