Sextic anharmonic oscillators and biconfluent Heun equations via differential Galois theory

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The one-dimensional Schrödinger equation with the sextic oscillator potential

We start considering the linear second order differential equation

$$V(r) = V_{-2}\, r^{-2} + A\, r^6 + B\, r^4 + C\, r^2, \quad r \in [0, \infty), A > 0, V_{-2} \geq 0 \tag{1}$$

By restricting the domain of definition to $r \in [0, \infty)$, and assign $V_{-2} = (\ell_d - 1)(\ell_d - 3)/4$, the problem can be interpreted as the radial component ($r \in [0, \infty)$) of a spherically symmetric potential with $\hat{h} = 2m = 1$ units:

$$-\frac{d^2\psi_{\ell_d}}{dr^2} + \left(\frac{(\ell_d - 1)(\ell_d - 3)}{4r^2} + A\, r^6 + B\, r^4 + C\, r^2\right)\psi_{\ell_d} = E^{\ell_d}_{A,B,C}\, \psi_{\ell_d}, \tag{2}$$

where $r \in [0, \infty)$, $\psi_{\ell_d}(0) = \psi_{\ell_d}(\infty) = 0$, $\ell_d = d + 2\,\ell$, $\ell = 0, 1, 2, \cdots$.

Setting $V_{-2} = 0$, the domain of definition of the sextic anharmonic oscillator potential can be extended to the full real axis, $r \in (-\infty, \infty)$. A case in which the bound-state wave function solutions of

$$-\frac{d^2\psi}{dr^2} + \left(A\, r^6 + B\, r^4 + C\, r^2\right)\psi = E_{A,B,C}\,\psi, \tag{3}$$

with $\quad r \in (-\infty, \infty), \quad \psi(\pm\,\infty) = 0$, have definite parity and the odd solutions that vanish at $r = 0$ correspond to the solution of the radial problem.

Link with Differential Galois Theory

We consider the family of second order trace-free linear differential equations,

$$\frac{d^2y}{dx^2} = L(x)y, \quad L(x) \in \mathbb{C}[x, x^{-1}] \tag{4}$$

where $L(x) \in \mathbb{C}[x, x^{-1}]$ is a monic Laurent polynomial

$$L(x) = \frac{\ell_{-r}}{x^r} + \ldots + \ell_0 + \ldots + \ell_{m-1}x^{m-1} + x^m$$

with $\ell_{-r} \neq 0$. We say that $L(x)$ has type (r, m). The space $\mathbb{M}_{(r,m)}$ of monic Laurent polynomials of type (r, m) is an affine algebraic variety $\mathbb{M}_{(r,m)} \simeq \mathbb{C}^* \times \mathbb{C}^{r+m-1}$. The purpose is to classify Picard-Vessiot integrable equations in the family (4) and to study how their Liouvillian solutions depend algebraically on the coefficients of $L(x)$ when it moves in the space $\mathbb{M}_{(r,m)}$. That is the *Liouvillian spectral set*.

Liouvillian Spectral Set of Schrödinger Equation

The Liouvillian *spectral set* $\mathbb{S}_{(r,m)} \subset \mathbb{M}_{(r,m)}$ is the set of monic Laurent polynomials of type (r, m) such that its corresponding Eq. (4) is Picard-Vessiot integrable.
Differential automorphism $\sigma : L \to L$:
i) automorphism of the field L,
ii) $\sigma d/dt = d/dt\sigma$.

Galois group of the equation (1):
$G := Gal(1) = Gal(L/K) = \{\sigma : L \to L : \sigma$ differential automorphism $\sigma_K = Id\}$
($\sim$ Galois group of a polynomial).
G is the transformation group preserving all the algebraic relations of u_{ij} with coefficients in K.

Theorem

G is a linear algebraic subgroup of $GL(m,)$.

(1) *integrable*: the extension L is obtained from K by a combination of algebraic extensions, quadratures and exponentials of quadratures.

Theorem

Eq. (1) is integrable $\Leftrightarrow$ G^0 is solvable.

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The asymptotic solution of the Schrödinger equation

We are now in a position to give the following result.

Theorem

The asymptotic solution of the Schrödinger equation

$$-\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\,\psi(x)$$

where $V(x)$ is a even polynomial, $V(x) = V(-x)$, of degree $4k+2,\ k = 1, 2, \cdots,$ assume the form

$$\psi(x) = \exp(-\phi(x))$$

where $\phi(x)$ is a even polynomial of degree $2k+2$ with a positive leading coefficient.

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Sextic harmonic oscillator and Heun Biconfluent Equation

We Recall that Biconfluent Heun Differential Equation (BHE) is

$$\frac{d^2y}{dx^2}+\left(-2x-\beta+\frac{1+\alpha}{x}\right)\frac{dy}{dx}+\left(\gamma-\alpha-2-\frac{(1+\alpha)\beta+\delta}{2x}\right)y=0 \tag{5}$$

where $\mathrm{BHE}(\alpha,\beta,\gamma,\delta;x)$ denote the solutions of (5). Due to is the radial part of 3D-SE, that is, $r>0$, we can set the change of variables

$$(r,\Psi)\mapsto\left(\frac{\sqrt{2}}{2}r^2,r^{\pm(\frac{k}{2}-1)-\frac{1}{2}}\exp\left(\frac{1}{4}r^4+\frac{\mu_1}{4}r^2\right)\Psi\right) \tag{6}$$

is equivalent to BHE (5), depending on $r>0$, with parameters given by

$$(\alpha,\beta,\gamma,\delta)=\left(\pm\left(\frac{k}{2}-1\right),\frac{\sqrt{2}}{2}\mu_1,\frac{1}{8}\mu_1^2-\frac{1}{2}\mu_2,-\frac{\sqrt{2}}{2}E^k_{\mu_1,\mu_2}\right) \tag{7}$$

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BHE and DGT

Integrability of Heun Biconfluent Differential Equation has been analyzed through Differential Galois theory by several authors.

The Liouvillian Spectral Set of BHE is non-empty, and therefore Schrödinger equation with sextic anharmonic oscillator is quasisolvable.

In our work (with Ismail and Saad) we were able to solve the d-dimensional Schrödinger equation for arbitrary μ_4 and μ_2 by computing the connection coefficients between the two fundamental sets of solutions at the original and the infinity for the biconfluent Heun equation. The method discussed there solve the eigenvalue problem of biconfluent heun equation and consequently the schrödinger equation associated with it. A simplified technique based on the Asymptotic iteration method was introduced to compute the series coefficients of the solutions to the Biconfluent Heun Equation.

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References

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