Applied Math II - Qualifier Portion 2016

Do 6 of 7 problems

1. a) Assume that A is an n by n symmetric matrix with eigenvalues ordered $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. Show that $\rho \leq \lambda_n$, where ρ is a Rayleigh quotient.

b) Give a reason why this is not necessarily true if A is not symmetric.

c) Assume A is symmetric. Prove the eigenvectors of A are orthogonal if they correspond to distinct eigenvalues (you may assume that we have already shown that the eigenvalues are real).

2. a) What can you say about diagonalizability if $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & -4 \end{bmatrix}$ (without doing any calculations) ?

b) Same for $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 1 & 1 & 4 \end{bmatrix}$.

c) Given A has eigenvalues 0, 1, 2, 3, and 4, what can you say about diagonalizability?

d) What are the eigenvalues and eigenvectors if $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \frac{\pi}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

e) For the matrix in part d), compute $cos(A)$. Give justification for how you compute this. (Hint: $cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots$).

3. a) Discuss convergence of a Krylov subspace method for computing eigenvalues. When will it converge fast and when slow? (Should have a picture with a polynomial drawn.)

b) If A has eigenvalues $1, 2, 3, \ldots n$, compare convergence of a Krylov method for computing the eigenvalue at 1 with a Krylov method for solving linear equations with A.

c) For a Krylov method for linear equations, which eigenvalue distribution is likely to give faster convergence and why:

i) $1, 2, 3, \ldots 1000$

ii) $0.1, 0.2, 0.3, \ldots 19.9, 20$?

4. a) Consider the simple wave equation $u_t = -au_x$, $u(x, 0) = q(x)$.

Show that $u(x,t) = q(x - at)$ is a solution.

b) Discuss how to solve this numerically (on an interval in space and an interval in time).

5. Consider this 1-D, elliptic boundary value problem:

$u'' + 2u' = 3x$
$u(0) = 3$
$u(1) = 0$

Find the finite difference solution with $h = \frac{1}{3}$. You may stop after setting up the 2 by 2 system of linear equations.

6. Consider this 1-D, elliptic boundary value problem:

$u'' + \delta(x - \frac{1}{2})u = 0$
$u'(0) = 2$
$u(1) = -1$

Find the finite element solution with $h = \frac{1}{2}$. (Use linear basis functions.)

7. Using the characterization of the Dirac delta function as the limit of impulse functions, show that $\int_0^\infty x\delta(x - 7)dx = 7$.