Solve FOUR of the following problems.

1. (a) Show that \( L(f)(x) = \int_0^x t^2 f(t) dt \) defines a continuous linear mapping from \( C([0, 1]) \) into itself, and find its operator norm.

(b) Is the mapping \( L \) defined above compact? Why or why not?

2. Let \( H \) be a Hilbert space with \( \{e_n\}_{n \geq 1} \subset H \) a sequence and \( \{\lambda_n\}_{n \geq 1} \) a sequence of scalars.

   (a) State what it means for \( \{e_n\}_{n \geq 1} \subset H \) to be a complete orthonormal sequence.

   (b) If \( \{e_n\} \) is a complete orthonormal sequence, show that there exists a unique linear operator \( T \) on \( H \) such that \( Te_n = \lambda_n e_n \).

   (c) Show that \( T \) is bounded iff \( \{\lambda_n\} \) is.

   (d) When \( T \) is bounded, what is its norm in terms of \( \{\lambda_n\} \)?

3. (a) State the Banach fixed point theorem.

(b) Define the (nonlinear!) operator \( T \) mapping \( C[0, 1] \) into itself by

\[
(Tu)(t) = \int_0^t (u(x))^2 dx,
\]

and show that, although \( T \) is not a contraction on the closed unit ball, it is on the closed ball of radius \( \frac{1}{4} \).

(c) Does there exist a continuous function \( u \) such that \( Tu = u \)?

4. Let \( V \) be an inner product space.

   (a) State what it means for a sequence to converge \textit{weakly} in \( V \).

   (b) Show that, if \( V \) is also finite-dimensional and Hilbert, that every weakly convergent sequence is strongly convergent.

   (c) Prove or give a counterexample: If \( x_n \rightharpoonup x \) and \( y_n \rightharpoonup y \) as \( n \to \infty \) in a Hilbert space, then \( (x_n, y_n) \to (x, y) \).
5. (a) State the Riesz Representation Theorem for a Hilbert space $H$ over $\mathbb{C}$.

(b) Because this theorem associates to each member of $H'$ a member of $H$, it defines a mapping $T$ from $H'$ into $H$. Show that the following properties hold for all $f, g \in H$ and all $\alpha \in \mathbb{C}$:

\[
T(f + g) = T(f) + T(g) \\
T(\alpha f) = \overline{\alpha}T(f) \\
\|T(f)\| = \|f\|
\]