Ph.D. Qualifying Exam

Applied Mathematics I

Solve FOUR of the following problems.

1. (a) Show that $L(f)(x) = \int_0^x t^2 f(t)dt$ defines a continuous linear mapping from $C([0,1])$ into itself, and find its operator norm.

   (b) Is the mapping L defined above compact? Why or why not?

2. Let H be a Hilbert space with $\{e_n\}_{n\geq 1} \subset H$ a sequence and $\{\lambda_n\}_{n\geq 1}$ a sequence of scalars.

   (a) State what it means for $\{e_n\}_{n\geq 1} \subset H$ to be a complete orthonormal sequence.

   (b) If $\{e_n\}$ is a complete orthonormal sequence, show that there exists a unique linear operator T on H such that $Te_n = \lambda_n e_n$.

   (c) Show that T is bouneded iff $\{\lambda_n\}$ is.

   (d) When T is bounded, what is its norm in terms of $\{\lambda_n\}$?

3. (a) State the Banach fixed point theorem.

   (b) Define the (nonlinear!) operator T mapping $C[0,1]$ into itself by

   $$(Tu)(t) = \int_0^t (u(x))^2 \, dx,$$

   and show that, although T is not a contraction on the closed unit ball, it is on the closed ball of radius $\frac{1}{4}$.

   (c) Does there exist a continuous function u such that $Tu = u$?

4. Let V be an inner product space.

   (a) State what it means for a sequence to converge *weakly* in V.

   (b) Show that, if V is also finite-dimensional and Hilbert, that every weakly convergent sequence is strongly convergent.

   (c) Prove or give a counterexample: If $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$ as $n \to \infty$ in a Hilbert space, then $(x_n, y_n) \to (x, y)$.

5. (a) State the Riesz Representation Theorem for a Hilbert space H over $\mathbb{C}$.

   (b) Because this theorem associates to each member of H' a member of H, it defines a mapping T from H' into H. Show that the following properties hold for all $f, g \in H$ and all $\alpha \in \mathbb{C}$:

$$\begin{aligned} T(f+g) &= T(f) + T(g) \\ T(\alpha f) &= \overline{\alpha} T(f) \\ \|T(f)\| &= \|f\| \end{aligned}$$