Topology **Ph.D. Qualifying Exam** **August 11, 2017**

(1) (3 points each) Complete each of the following definitions.

(a) An *ordinal* is ...

(b) A *basis* for a topology ...

(c) A topological space is *separable* ...

(d) A topological space is *normal* ...

(e) Suppose $\{X_a : a \in A\}$ is a collection of topological spaces. The *Cartesian product* $\prod_A X_a$ is ...

(f) Suppose x_0 is a point of a topological space X. The *fundamental group* of X relative to the *base point* x_0 is ...

(g) A subspace A of X is said to be a *deformation retract* of X if and only if ...

(h) The *singular chain group* of a nonempty topological space X in dimension p ...

(i) For each positive integer p, the *boundary operator* on a topological space X in dimension p ...

(j) A *p-cycle* of a topological space X ...

(2) (3 points each) Give an example of each of the following, or state that no such example exists. You need not show any work.

(a) a T_1 space that is not Hausdorff

(b) two sets, H and K, such that $H^\circ \cap K^\circ \neq (H \cap K)^\circ$.

(c) a compact set that is not closed.

(d) a connected set that fails to be path connected.

(e) a space X and two points, x_0 and x_0' of X such that $\Pi_1(X, x_0)$ and $\Pi_1(X, x_0')$ are not isomorphic.

(f) a covering map for S^1 that is not a homeomorphism.

(g) two spaces that are homotopy equivalent, but neither is a deformation retraction of the other.

(h) a space X with a path component A and a point a_0 of A such that $\Pi_1(X, a_0)$ and $\Pi_1(A, a_0)$ are not isomorphic.

(i) A space X such that $H_1(X)$ is nonabelian.

(j) A space X for which $H_1(X)$ is trivial, but $H_2(X)$ is nontrivial.

(3) (10 points each) Prove three of the following.

(a) Suppose $G = \{X_a : a \in A\}$ is a collection of topological spaces, X is a topological space, and f is a function from X into $\prod_A X_a$. If $\pi_a \circ f$ is continuous for each a in A, then f is continuous.

(b) Every separable metric space is second countable.

(c) If A is a connected subset of M, and M is the union of two mutually separated sets, then A is a subset of one of them.

(d) The intersection of a countable collection of dense open subsets of a locally compact Hausdorff space is dense.

(4) (10 points each) Prove three of the following.

(a) Suppose f is a path in a space X from x_0 to x_1 and f' is a path in X from x_1 to x_2. If g and g' are paths such that $f \simeq_p g$ and $f' \simeq_p g'$, then $f * f' \simeq_p g * g'$.

(b) If $h : (X, x_0) \to (Y, y_0)$ is a map, then h_* is a homomorphism.

(c) If $f : X \to Y$ is a map, then the chain map $(f_\#)_p$ maps p-boundaries in X to p-boundaries in Y for each p.

(d) If (X, A) is a topological pair, and the inclusion map $i : A \to X$ is a homotopy equivalence, then $H_q(X, A) \cong 0$ for each q. (Assume only the Axioms of Homology and their consequences.)